Chapter 1

Antennas

1.1 Calculating Radiated Fields

We have talked a lot about waves: how they propagation, how they behave at interfaces, and how to guide them. We know need to learn how to generate these waves. The basic principle is: *currents radiate fields.*

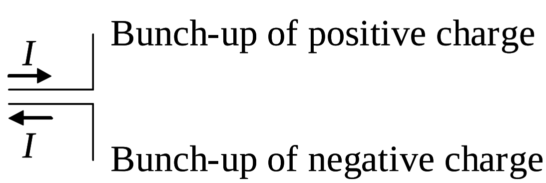
The question is: how do we determine the fields if we know the currents? Let's go back to our wave equation for $\overline{E}$, but this time allow for the presence of currents.

$$\nabla \times (\nabla \times \overline{E}) = -j\omega\mu\nabla \times \overline{H} = -j\omega\mu(j\omega\epsilon\overline{E} + \overline{J}) = \omega^2\mu\epsilon\overline{E} - j\omega\mu\overline{J}$$

We can now use:

$$\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) - \nabla^2\overline{E}$$

Normally, we let $\nabla \cdot \overline{E} = 0$ since $\nabla \cdot \overline{D} = 0$ in a charge-free region. However, we typically create radiation by injecting current into a metal structure. Charge can "bunch" up, resulting in an excess of charge so that $\nabla \cdot \overline{D} \neq 0$. This makes the wave equation too difficult to solve. We therefore resort to an alternate procedure.



We recall that we used the magnetic vector potential $\overline{A}$ (units Wb/m) in magnetostatics to simplify the analysis. The same approach can be used here. We recall that

$$\overline{B} = \nabla \times \overline{A}$$

which naturally satisfies $\nabla \cdot \overline{B} = 0$ since $\nabla \cdot (\nabla \times \overline{A}) = 0$ for any arbitrary vector $\overline{A}$. Let's go through this for dynamic fields.

$$\nabla \times \overline{E} = -j\omega\overline{B} = -j\omega\nabla \times \overline{A}$$

so

$$\nabla \times \overline{E} + j\omega \nabla \times \overline{A} = \nabla \times (\overline{E} + j\omega \overline{A}) = 0$$

We now define

$$\overline{E} + j\omega \overline{A} = -\nabla \phi$$

where ϕ is the scalar electric potential for dynamics. We make this definition because there is an identity $\nabla \times \nabla \phi = 0$ for any scalar function ϕ. The minus sign is simply there for consistency with electrostatic electric potential.

Now, using Ampere's law:

$$\nabla \times \overline{H} = j\omega\epsilon \overline{E} + \overline{J}$$

Since

$$\overline{H} = \frac{\overline{B}}{\mu} = \frac{1}{\mu} \nabla \times \overline{A}$$

$$\begin{aligned}
\frac{1}{\mu} \nabla \times \nabla \times \overline{A} &= \overline{J} + j\omega\epsilon \overline{E} = \overline{J} + j\omega\epsilon(-j\omega \overline{A} - \nabla \phi) \\
\nabla(\nabla \cdot \overline{A}) - \nabla^2 \overline{A} &= \mu \overline{J} + \omega^2 \mu\epsilon \overline{A} - j\omega\mu\epsilon \nabla \phi
\end{aligned}$$

Now, so far we have only specified $\nabla \times \overline{A}$, which does not uniquely specify $\overline{A}$. We need to specify $\nabla \cdot \overline{A}$ as well. A good choice is:

$$\nabla \cdot \overline{A} = -j\omega\mu\epsilon\phi$$

We call this the *Lorentz gauge*. We can take the gradient to obtain $\nabla(\nabla \cdot \overline{A}) = -j\omega\mu\epsilon\nabla\phi$. Then,

$$-\nabla^2 \overline{A} = \mu \overline{J} + \omega^2 \mu\epsilon \overline{A} = \mu \overline{J} + k^2 \overline{A}$$

or

$$(\nabla^2 + k^2)\overline{A} = -\mu \overline{J}$$

This is a differential equation that can be solved. The resulting solution gives the magnetic vector potential $\overline{A}$ given a current distribution $\overline{J}$. Once we know $\overline{A}$, we can obtain $\overline{H}$ and $\overline{E}$. There is a very elegant way to solve this equation generally, but this approach is a bit too complicated for this class. The end result of this analysis is that for a current density $\overline{J}(\overline{r}')$:

$$\overline{A}(\overline{R}) = \frac{\mu}{4\pi} \int_V \overline{J}(\overline{R}') \frac{e^{-jk|\overline{R} - \overline{R}'|}}{|\overline{R} - \overline{R}'|} dV'$$

Remember that $\overline{R}$ when used as the argument of a function is simply a surrogate for the position. For example, in Cartesian coordinates we could write $\overline{J}(\overline{R}') = \overline{J}(x', y', z')$. We have the primed variables which are the integration or *source* coordinates. The unprimed variables represent *observation* variables.

If you have had 380, you can see that this result is a convolution, where $\overline{R}$ is like t and $\overline{R}'$ is like τ. $\overline{A}(\overline{R}) = \overline{J}(\overline{R}) * g(\overline{R})$ where $g(\overline{R})$ plays the role of the impulse response.

1.2 Hertzian Dipole

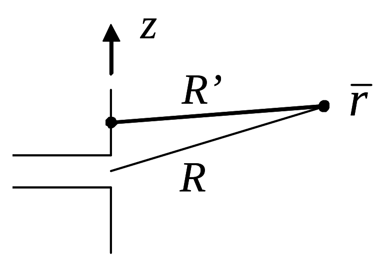
1.2.1 General Derivation

Let's show how to apply this. We start with the simplest case. Suppose we have a very short dipole of length ℓ. Since the dipole length is short compared to the wavlength, the current is constant over the length of the antenna. The current density can be written as

$$\overline{J}(\overline{r}') = \overline{J}(x', y', z') = I_o \hat{z} \delta(x') \delta(y') \qquad -\ell/2 \leq z' \leq \ell/2.$$

Plugging this into the equation for the magnetic vector potential yields

$$\overline{A}(\overline{R}) = \frac{\mu_0}{4\pi} \int_{-\ell/2}^{\ell/2} I_o \hat{z} \frac{e^{-jk|\overline{R}-\overline{R}'|}}{|\overline{R}-\overline{R}'|} dz'$$



Notice that for a very short dipole, $|\overline{R} - \overline{R}'| \approx R$.

$$\overline{A}(\overline{R}) = \frac{\mu_0}{4\pi} \int_{-\ell/2}^{\ell/2} I_o \hat{z} \frac{e^{-jkR}}{R} dz' = \frac{\mu_0}{4\pi} I_o \ell \hat{z} \frac{e^{-jkR}}{R}$$

We call this the **Hertzian dipole**.

Typically, we are interested in determining the field a fixed distance away from the origin. So, instead of using Cartesian coordinates, it's easier to use spherical coordinates:

$$\hat{z} = \hat{R} \cos\theta - \hat{\theta} \sin\theta$$

$$\overline{A} = (\hat{R} \cos\theta - \hat{\theta} \sin\theta) \frac{\mu_0}{4\pi} I_o \ell \frac{e^{-jkR}}{R}$$

Now, using:

$$\begin{aligned}
\overline{H} &= \frac{1}{\mu_0} \nabla \times \overline{A} \\
&= \hat{\phi} \frac{I_o \ell}{4\pi} \left[jk + \frac{1}{R} \right] \sin\theta \frac{e^{-jkR}}{R} \\
\overline{E} &= \frac{1}{j\omega\epsilon_0} \nabla \times \overline{H} \\
&= \hat{R} \frac{2\eta_0 I_o \ell}{4\pi} \left[1 - \frac{j}{kR} \right] \cos\theta \frac{e^{-jkR}}{R^2} + \hat{\theta} \frac{\eta_0 I_o \ell}{4\pi} \left[jk + \frac{1}{R} - \frac{j}{kR^2} \right] \sin\theta \frac{e^{-jkR}}{R}
\end{aligned}$$

The power radiating from the antenna would be the $\hat{R}$ component of $\overline{S}$.

$$\begin{aligned}
S_R &= \overline{E} \times \overline{H}^* \cdot \hat{R} \\
&= \left\{ \hat{\theta} \frac{\eta_0 I_o \ell}{4\pi} \left[jk + \frac{1}{R} - \frac{j}{kR^2} \right] \sin\theta \frac{e^{-jkR}}{R} \right\} \times \left\{ \hat{\phi} \frac{I_o \ell}{4\pi} \left[jk + \frac{1}{R} \right] \sin\theta \frac{e^{-jkR}}{R} \right\}^* \cdot \hat{R} \\
&= \eta_0 \left| \frac{I_o \ell}{4\pi} \right|^2 \frac{\sin^2\theta}{R^2} \left\{ k^2 + \frac{jk}{R} - \frac{jk}{R} + \frac{1}{R^2} - \frac{1}{R^2} - \frac{j}{kR^3} \right\} \\
&= \eta_0 \left| \frac{I_o \ell}{4\pi R} \right|^2 \sin^2\theta \left\{ k^2 - \frac{j}{kR^3} \right\}
\end{aligned}$$

So, the real power decays as $1/R^2$ as expected. There is some reactive power stored near the dipole which decays as $1/R^5$.

$$S_{av,R} = \frac{\eta_0}{2} \left| \frac{kI_o \ell}{4\pi R} \right|^2 \sin^2\theta = S_o \sin^2\theta$$

We are generally interested in field behaviors far from the dipole. So, we only keep the dominant terms for large R. This means we neglect any field terms that decay as $1/R^2$, $1/R^3$.

$$\begin{aligned}
\overline{E}_{\text{ff}} &= \hat{\theta} \frac{\eta_0}{4\pi} I_o \ell jk \frac{e^{-jkR}}{R} \sin\theta \\
\overline{H}_{\text{ff}} &= \hat{\phi} \frac{1}{4\pi} I_o \ell jk \frac{e^{-jkR}}{R} \sin\theta
\end{aligned}$$

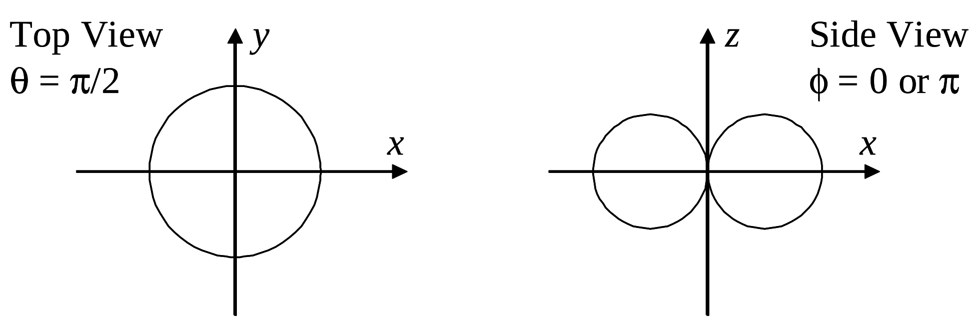
We see that

$$\overline{H}_{\text{ff}} = \frac{1}{\eta_0} \hat{k} \times \overline{E}_{\text{ff}}$$

just like for plane waves. This confirms what we have been saying that a spherical wave behaves like a plane wave when the sphere radius is large.

$$\overline{S}_{av,ff} = \hat{R} \frac{\eta_0}{2} \left| \frac{kI_o \ell}{4\pi R} \right|^2 \sin^2\theta = \hat{R} S_o \sin^2\theta$$

For a fixed radius the power density $\overline{S}_{av,ff}$ varies with θ but not with ϕ. We get the donut pattern shown below.



The power density of the Hertzian dipole decays with distance $1/R^2$.

1.3 General Antenna Parameters

1.3.1 Antenna Pattern

The antenna pattern is a functional representation of how the radiated power density (in the far-field) varies with position. It is defined as:

$$F(\theta, \phi) = \frac{\overline{S}_{av,ff} \cdot \hat{R}}{(\overline{S}_{av,ff} \cdot \hat{R})_{max}}$$

The antenna pattern is often shown in dB and many times in polar coordinates. The main parameters that are often extracted from the antenna pattern are the beam width and the maximum side lobe power.

Let's examine the antenna pattern a little more. To aid in this, we define the solid angle. In spherical coordinates:

$$dA = R^2 \sin\theta d\theta d\phi$$

We define the differential solid angle as

$$d\Omega = \frac{dA}{R^2} = \sin\theta d\theta d\phi$$

We have shown before that

$$\int_0^{2\pi} \int_0^{\pi} d\Omega = 4\pi$$

So, full solid angle is 4π (units = steradians)

Now, the power radiated by the antenna is:

$$\begin{aligned} P_{rad} &= \int_0^{2\pi} \int_0^{\pi} \overline{S}_{av}(\theta, \phi) \cdot \hat{R} R^2 \sin\theta d\theta d\phi \\ &= R^2 \int_0^{2\pi} \int_0^{\pi} S_{av,R}(\theta, \phi) d\Omega \\ &= R^2 S_{max} \int_0^{2\pi} \int_0^{\pi} F(\theta, \phi) d\Omega \end{aligned}$$

Often, we plot the radiation pattern in dB:

$$F_{dB} = 10 \log F$$

This allows us to clearly see nulls in the pattern. When we plot this, we define:

1. **Main Lobe:** Angular region where most of the energy is transmitted
2. **Side Lobes:** Smaller lobes of energy transmission
3. **Elevation Plane:** Plane for a constant value of ϕ
4. **Azimuth Plane:** Plane for $\theta = 90°$
5. **Pattern Beamwidth:** Angular extent of main lobe between two angles at which $|F(\theta, \phi)|$ is half its peak value (-3 dB)

1.3.2 Directivity

The directivity characterizes the angular extent of the transmitted beam. A high directivity means that the power is confined to a smaller angular region. An antenna with a high directivity has good power confinement but requires more accurate pointing.

The directivity is defined as the maximum radiation pattern over its average as given by

$$D = \frac{F_{max}}{F_{av}} = \frac{1}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)d\Omega}$$

Note that:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)d\Omega = \frac{P_{rad}}{4\pi R^2 S_{max}}$$

So the directivity can also be written as

$$D = \frac{S_{max}}{P_{rad}/4\pi R^2}$$

Therefore, the directivity D represents the maximum power density over the power density that would occur if all the power were radiated equally in all directions.

$$D = \frac{S_{max}}{S_{\text{isotropic radiator}}}$$

1.3.3 Gain

Let P_t be the total power supplied to the antenna from the transmitter. The antenna radiation efficiency is:

$$\text{Radiation Efficiency: } \xi = \frac{P_{rad}}{P_t}$$

$$\text{Gain: } G = \frac{S_{max}}{P_t/4\pi R^2} = \frac{S_{max}}{P_{rad}/\xi 4\pi R^2} = \xi D$$

So, the gain accounts for losses in the antenna (or possibly impedance mismatches between the generator and the antenna), while directivity does not.

1.3.4 Radiation Resistance

To the transmission line feeding the antenna, the antenna is merely an impedance. In fact, antennas are simply matching devices between the transmission line and free space. We define a radiation resistance R_{rad} which relates the radiated power to the antenna driving current as:

$$P_{rad} = \frac{1}{2}I_o^2 R_{rad}$$

The ohmic loss in the antenna represented by resistance R_{loss} is:

$$P_{loss} = \frac{1}{2}I_o^2 R_{loss}$$

The total power delivered to the antenna is therefore:

$$P_t = P_{rad} + P_{loss} = \frac{1}{2} I_o^2 (R_{rad} + R_{loss})$$

We can therefore express the radiation efficiency as:

$$\xi = \frac{P_{rad}}{P_t} = \frac{R_{rad}}{R_{rad} + R_{loss}}$$

Since $R_{rad} = 2P_{rad}/I_o^2$, for a Hertzian dipole we have:

$$R_{rad} = \eta_0 \frac{(kI_o^2 \ell)^2}{12\pi} \frac{2}{I_o^2} = \eta_0 \frac{(k\ell)^2}{6\pi}$$

Since $\eta_0 = 120\pi$, $k = 2\pi/\lambda$:

$$R_{rad} = \frac{120\pi}{6\pi} \left(\frac{2\pi}{\lambda} \ell \right)^2 = 80 \left(\frac{\pi \ell}{\lambda} \right)^2$$

For very short dipoles ($\ell \to 0$), the radiation resistance is very small, which means it is hard to radiate real power with a short dipole.

If the Hertzian dipole length is

$$\ell = \frac{\lambda}{50}$$

then the radiation resistance is

$$R_{rad} = 80 \left(\frac{pi}{50} \right)^2 \ell = 0.3\Omega$$

Example

A satellite antenna is designed to illuminate the continental US. What is the antenna directivity?

Assume the antenna pattern is 1 inside of some cone and 0 outside.

Assume that the distance between the satellite and the earth is $L = 40,000km$ and that the diameter of the continental US is $2r = 2,500mi$ or $2r = 4 \times 10^6 m$.

The antenna illuminates a cone with an angle of

$$\tan\theta = \frac{r}{L} = \frac{2 \times 10^6}{4 \times 10^7}$$

$$\theta = 2.86^o$$

The antenna pattern is

$$F = \begin{cases} 1 & 0 < \theta < 2.86^o \\ 0 & \text{else} \end{cases} \tag{1.1}$$

$$D = \frac{1}{F_{av}} \tag{1.2}$$

$$
\begin{aligned}
F_{av} &= \frac{1}{4\pi R^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2.86^o} R^2 \sin\theta d\theta d\phi &(1.3)\\
&= \frac{1}{4\pi}(2\pi)\cos\theta|_0^{2.86^o} &(1.4)\\
&= \frac{1}{2}\left[1-\cos\left(2.86^o\right)\right] &(1.5)\\
&= 6.2\times 10^{-4} &(1.6)
\end{aligned}
$$

$$
D = \frac{1}{6.2\times 10^{-4}} = 1606 \qquad (1.7)
$$

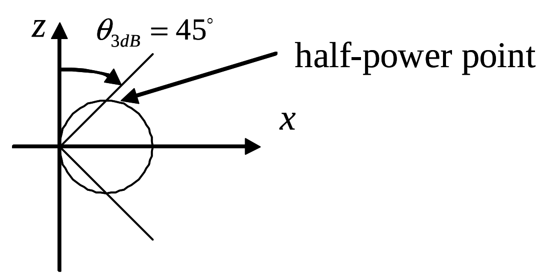
1.4 Hertzian Dipole

Let's examine the antenna parameters for the Hertzian dipole.

$$
\begin{aligned}
\overline{S}_{av,ff} &= \hat{R}\frac{\eta_0}{2}\left|\frac{kI_o\ell}{4\pi R}\right|^2 \sin^2\theta \\
F(\theta,\phi) &= \sin^2\theta
\end{aligned}
$$

The 3 dB beamwidth is:

$$
\begin{aligned}
\sin^2\theta_{3dB} &= \frac{1}{2} \\
\theta_{3dB} &= 45^\circ \\
\text{Beamwidth} &= \beta = 90^\circ
\end{aligned}
$$



The direction of maximum radiation occurs at $\theta = 90^\circ$ or perpendicular to the dipole.

$$
\begin{aligned}
S_{max} &= \frac{\eta_0}{2}\left(\frac{kI_o\ell}{4\pi R}\right)^2 \\
P_{rad} &= R^2 S_{max}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} F(\theta,\phi)\sin\theta d\theta d\phi \\
&= \frac{\eta_0}{2}\left(\frac{kI_o\ell}{4\pi}\right)^2\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin^3\theta d\theta d\phi \\
&= \frac{\eta_0}{2}\left(\frac{kI_o\ell}{4\pi}\right)^2 2\pi\int_{\theta=0}^{\pi}\sin^3\theta d\theta d\phi
\end{aligned}
$$

We need to integrate $\sin^3\theta$. Here is the basic procedure used. Substitute $\sin^2\theta = 1-\cos^2\theta$ to get

$$\begin{aligned} \int \sin^3\theta d\theta &= \int \left(1-\cos^2\theta\right)\sin\theta d\theta && (1.8)\\ &= \int \sin\theta d\theta - \int \cos^2\theta \sin\theta d\theta && (1.9) \end{aligned}$$

Do a u substitution for the cos term as given by

$$u = \cos\theta \quad (1.10)$$

$$du = -\sin\theta d\theta \quad (1.11)$$

The integral then becomes

$$\begin{aligned} \int \sin^3\theta d\theta &= \int \sin\theta d\theta - \int u^2 \frac{\sin\theta}{-\sin\theta} du && (1.12)\\ &= -\cos\theta + \frac{1}{3}u^3 && (1.13)\\ &= -\cos\theta + \frac{1}{3}\cos^3\theta && (1.14) \end{aligned}$$

Evaluating the integral becomes

$$\begin{aligned} \int_{\theta=0}^{\pi} \sin^3\theta d\theta &= \frac{1}{3}\left[\cos(\pi)-\cos(0)\right] - \left[\cos(\pi)-\cos(0)\right] && (1.15)\\ &= \left(2-\frac{2}{3}\right] && (1.16)\\ &= \frac{4}{3} && (1.17) \end{aligned}$$

$$\begin{aligned} \int_{0}^{\pi} \sin^3\theta d\theta &= -\frac{1}{3}\left(2+\sin^2\theta\right)\cos\theta\Big|_{\theta=0}^{\theta=\pi} && (1.18)\\ &= \left[-\frac{1}{3}(2+0)\ (-1)\right] - \left[-\frac{1}{3}(2+0)\ (1)\right] && (1.19)\\ &= \frac{1}{3}(2) + \frac{1}{3}(2) && (1.20)\\ &= \frac{4}{3} && (1.21) \end{aligned}$$

The resulting total radiated power is

$$\begin{aligned} P_{rad} &= \frac{\eta_0}{2}\left(\frac{kI_o\ell}{4\pi}\right)^2 (2\pi)\left(\frac{4}{3}\right) && (1.22)\\ &= \frac{\eta_0\left(kI_o\ell\right)^2}{12\pi} && (1.23) \end{aligned}$$

$$
\begin{aligned}
D &= \frac{F_{max}}{F_{av}} && (1.24)\\
&= \frac{1}{\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin^3\theta d\theta d\phi} && (1.25)\\
&= \frac{1}{\frac{1}{4\pi}\frac{8\pi}{3}} && (1.26)\\
&= 1.5 && (1.27)
\end{aligned}
$$

1.5 Dipole Antennas

The Hertzian dipole is great because it is easy to formulate the fields for this antenna. However, it is impractical because we cannot effectively radiate power with such an antenna (the radiation resistance is small). The analysis that we used assumes that the current along the dipole is constant. However, for dipoles of a practical length (say a half wavelength), the current is not constant along the dipole, and therefore our analysis is incorrect. We therefore desire to examine this more practical antenna structure.

Before we can do this analysis, however, we need to make some simplifications to our integral for $\overline{A}$. For realistic currents, we generally cannot perform the integration to compute $\overline{A}$. However, since we are typically interested in the far-fields, we can make a far-field approximation to the integral.

Let $\hat{R}$ be the unit vector in the direction of the observation vector $\overline{r}$. For a point $\overline{r}$ very far from the source point $\overline{r}'$, we can approximate the value

$$
|\overline{r}-\overline{r}'| \approx R - \hat{R}\cdot\overline{r}' \qquad (1.28)
$$

So, for the phase term in our Green's function, we can write

$$
e^{-jk|\overline{r}-\overline{r}'|} \approx e^{-jkR}e^{jk\hat{R}\cdot\overline{r}'} \qquad (1.29)
$$

For the magnitude, we can simplify this expression even further by neglecting the term $\hat{R}\cdot\overline{r}'$ to write

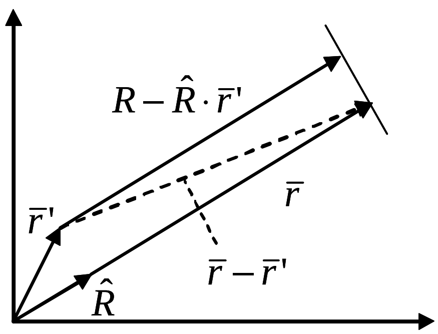
$$
\frac{1}{|\overline{r}-\overline{r}'|} \approx \frac{1}{R} \qquad (1.30)
$$

We therefore have our far-field approximate form of the magnetic vector potential given as

$$
\overline{A}_{\text{ff}}(\overline{r}) = \frac{\mu}{4\pi}\frac{e^{-jkR}}{R}\int \overline{J}(\overline{r}')e^{jk\hat{R}\cdot\overline{r}'} \qquad (1.31)
$$

Furthermore, when we take the curl of $\overline{A}_{\text{ff}}$ to obtain the magnetic and electric fields, we neglect any terms that come from this expression that decay faster than $1/R$ (i.e. terms that behave as $1/R^2$, $1/R^3$, etc. This simplification leads to the forms

$$
\begin{aligned}
\overline{B}_{\text{ff}} &= \nabla\times\overline{A}_{\text{ff}} \approx -jk\hat{R}\times\overline{A}_{\text{ff}}\\
\overline{E}_{\text{ff}} &= \frac{1}{j\omega\epsilon}\nabla\times\overline{H}_{\text{ff}} \approx -\frac{jk}{j\omega\epsilon}\hat{R}\times\overline{H}_{\text{ff}} \approx j\omega\hat{R}\times(\hat{R}\times\overline{A}_{\text{ff}})
\end{aligned}
$$



We can now do the integration for a half-wavelength dipole. A reasonable approximation for the current on a dipole is a sinusoid that goes to zero at the ends of the dipole wires, or $\overline{J}(\overline{r}') = \hat{z}I_o\delta(x')\delta(y')\cos(kz')$, $-\lambda/4 \leq z' \leq \lambda/4$. Then

$$\begin{aligned}
\overline{A}_{\text{ff}}(\overline{r}) &= \frac{\mu_0}{4\pi}\frac{e^{-jkR}}{R}\int_{-\lambda/4}^{\lambda/4} \hat{z}I_o\cos(kz')e^{jkz'\cos\theta}dz' \\
&= \hat{z}\frac{\mu_0}{8\pi}\frac{e^{-jkR}}{R}I_o\int_{-\lambda/4}^{\lambda/4}\left[e^{jkz'(\cos\theta+1)}+e^{jkz'(\cos\theta-1)}\right]dz' \\
&= (\hat{R}\cos\theta-\hat{\theta}\sin\theta)\frac{\mu_0}{2k\pi}\frac{e^{-jkR}}{R}I_o\frac{\cos\left[\pi/2\cos\theta\right]}{\sin^2\theta} \\
\overline{H}_{\text{ff}}(\overline{r}) &= -\frac{jk}{\mu_0}\hat{R}\times\overline{A}_{\text{ff}}(\overline{r}) = \hat{\phi}\frac{jI_o}{2\pi}\frac{e^{-jkR}}{R}\frac{\cos\left[\pi/2\cos\theta\right]}{\sin\theta} \\
\overline{E}_{\text{ff}} &= j\omega\hat{R}\times(\hat{R}\times\overline{A}_{\text{ff}}) = \hat{\theta}\frac{j\eta_0 I_o}{2\pi}\frac{e^{-jkR}}{R}\frac{\cos\left[\pi/2\cos\theta\right]}{\sin\theta}
\end{aligned}$$

The time-average Poynting vector is:

$$S_{av,R} = \frac{|\overline{E}_{\text{ff}}|^2}{2\eta_0} = \frac{\eta_0|I_o|^2}{8(\pi R)^2}\left\{\frac{\cos\left[\pi/2\cos\theta\right]}{\sin\theta}\right\}^2$$

This Poynting vector is maximum at $\theta = \pi/2$ with the maximum being

$$S_{max} = \frac{\eta_0|I_o|^2}{8(\pi R)^2}$$

Therefore, the radiation pattern is:

$$F(\theta) = \left\{\frac{\cos\left[\pi/2\cos\theta\right]}{\sin\theta}\right\}^2$$

With this radiation pattern, we can determine:

Radiated Power:	$P_{rad} = 36.6|I_o|^2$
Directivity:	$D = 1.64$
Radiation Resistance:	$R_{rad} = 73\Omega$

1.6 Receiving

Antennas are also used for capturing energy from an incident wave and converting it into power.

The power collected by the receiving antenna depends on the power density of the the incident wave and the effective collecting area of the antenna as given by

$$P_{rec} = S_i \, A_e, \tag{1.32}$$

where S_i is the power density of the incident wave, P_{rec} is the power collected by the receiver, and A_e is the effective collecting area of the receiving antenna.

The basic derivation process is:

1. calculate the amount of power collected by a Hertzian dipole
2. relate this to an effective area the collected power
3. generalize to an arbitrary antenna by relating the effective area to the directivity

These following derivation assume that (1) the antenna is impedance matched to the waveguide and (2) the antenna loss is low $R_{loss} \ll R_{rad}$.

The first step is to calculate the collected power for a given incident power density. The load is matched to the antenna $Z_L = Z_{in}^*$. The load current is thus given by

$$I_L = \frac{V_{oc}}{Z_{in} + Z_L} = \frac{V_{oc}}{2R_{rad}}$$

$$\begin{aligned} P_{rec} &= \frac{1}{2} |I_L|^2 R_{rad} && (1.33) \\ &= \frac{1}{2} \frac{|V_{oc}|}{(2R_{rad})^2} R_{rad} = \frac{|V_{oc}|^2}{8R_{rad}} && (1.34) \end{aligned}$$

The power density is related to the incident electric field as given by

$$S_i = \frac{|E_i|^2}{2\eta_o} = \frac{|E_i|^2}{240\pi}$$

The effective area of the antenna is

$$A_e = \frac{P_{rec}}{S_i} = \frac{|V_{oc}|^2}{8R_{rad}} \frac{240\pi}{|E_i|^2} = \left| \frac{V_{oc}}{E_i} \right|^2 \frac{30\pi}{R_{rad}} \tag{1.35}$$

For a Hertzian dipole the field is constant across the antenna aperture resulting in

$$V_{oc} = E_i \, \ell$$

and

$$R_{rad} = 80\pi^2 \left(\frac{\ell}{\lambda} \right)^2 .$$

The effective area can then be calculated to be

$$A_e = \left| \frac{E_i \, \ell}{E_i} \right|^2 \frac{30\pi}{80\pi^2} \left(\frac{\lambda}{\ell} \right)^2 \tag{1.36}$$

$$= \frac{3\lambda^2}{8\pi} \tag{1.37}$$

Relating this to the gain of a Hertzian dipole results in

$$A_e = \frac{\lambda^2 G}{4\pi}, \tag{1.38}$$

which can be applied to any antenna.

Now we want to couple the transmitting and receiveing antennas together to get a complete link. Start with calculating the the power density produced by the transmitting antenna as given by

$$G_t = \frac{\text{Power Density}}{\text{Power density of an isotropic radiator}} \tag{1.39}$$

$$= \frac{S_i}{\frac{P_t}{4\pi R^2}} \tag{1.40}$$

$$= S_i \left(\frac{4\pi R^2}{P_t} \right) \tag{1.41}$$

$$S_i = G_t \left(\frac{P_t}{4\pi R^2} \right) \tag{1.42}$$

Now determine the power collected by the receiving antenna as given by

$$P_{rec} = S_i A_r. \tag{1.43}$$

Relate the effective area to the antenna gain to get

$$P_{rec} = S_i G_r \left(\frac{\lambda^2}{4\pi} \right). \tag{1.44}$$

Now plug in the equation relating the power density to the transmitting antenna to give

$$P_{rec} = \left(G_t \frac{P_t}{4\pi R^2} \right) G_r \left(\frac{\lambda^2}{4\pi} \right). \tag{1.45}$$

Resulting in Friis transmission formula, which is given by

$$\frac{P_{rec}}{P_t} = G_t \, G_r \, \left(\frac{\lambda}{4\pi R} \right)^2. \tag{1.46}$$

Example

A satellite to ground link is established for satellite TV with the following system parameters:

- $P_t = 100W$
- $L = 40,000km$
- minimum detectable power $P_{rec} = 1pW$
- the antennas are completely lossless

The effective area of a dish antenna is approximately equal to the area of the dish. In order to keep the price down the transmitting antenna is chosen to be 4 times larger than the antenna on the ground $A_t = 4\ A_r$. What is the diameter of the receiving antenna?

$$\frac{P_{rec}}{P_t} = G_t G_r \left(\frac{\lambda}{4\pi R}\right)^2 \tag{1.47}$$

$$\frac{10^{-12}}{100} = A_r\ A_t \left(\frac{4\pi}{\lambda^2}\right)^2 \left(\frac{\lambda}{4\pi R}\right)^2 \tag{1.48}$$

$$10^{-14} = A_r\ A_t \left(\frac{1}{\lambda R}\right)^2 \tag{1.49}$$

Since we chose $A_t = 4A_r$, we get

$$10^{-14} = 4A_r^2 \left(\frac{1}{\lambda R}\right)^2 \tag{1.50}$$

$$10^{-14} = 4A_r^2 \left(\frac{1}{\lambda R}\right)^2 \tag{1.51}$$

$$A_r = \sqrt{\frac{10^{-14}}{4}}\ 4 \times 10^7 \lambda \tag{1.52}$$

$$= 2\ \lambda \tag{1.53}$$

If the frequency is C-band (f=4GHz) then the antenna diameter is $d = 0.4m$.

1.7 Antenna Arrays

There are a variety of different benefits of antenna arrays. Here are a few of the most common advantages:

- Increasing the directivity of a simple antenna
- Concentrating the radiated power where you want it
- Eliminating the radiated power (or received power) where you don't want it
- Electronically steering the direction of a highly directional antenna

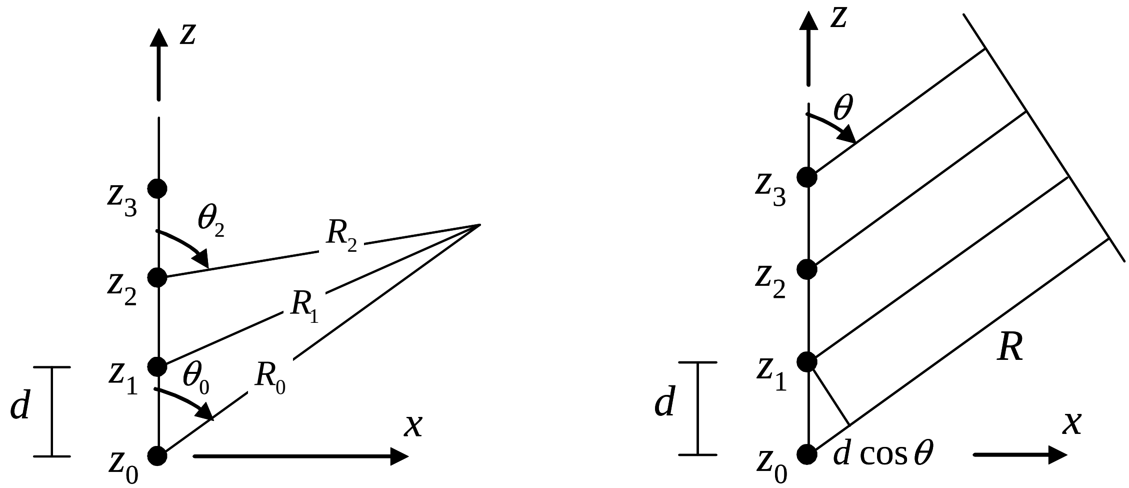
For our Hertzian dipole antenna:

$$\overline{E}_{\text{ff}} = \hat{\theta}\frac{\eta_0}{4\pi} I_o \ell jk \frac{e^{-jkR}}{R}\sin\theta = \hat{\theta} E_0 \frac{e^{-jkR}}{R}\sin\theta$$

Suppose we now have several such antennas, each having a different excitation current (magnitude and phase) arranged in a line. We call this an **antenna array**. We will generalize our far-field electric field for a single element so that our analysis can apply to any element type (not just the Hertzian dipole). The ith element will have electric field

$$\overline{E}_i(R_i, \theta_i, \phi_i) = A_i \frac{e^{-jkR_i}}{R_i} \overline{f}_e(\theta_i, \phi_i)$$

We have broken down this single element field into a current weight ($A_i = a_i e^{j\psi_i}$), a spherical wave factor (e^{-jkR_i}/R_i), and a vector function that depends only on the observation angles ($\overline{f}_e(\theta_i, \phi_i)$. The distance R_i is of course the distance from the ith element to the observation point. The angles have a similar definition, as shown.



The total field for this array is

$$\overline{E} = \sum_{i=0}^{N-1} A_i \frac{e^{-jkR_i}}{R_i} \overline{f}_e(\theta_i, \phi_i)$$

We now go back to our far-field assumption. From the figure, we can see that the far-field approximation suggests $\theta_i = \theta$ and $\phi_i = \phi$ for all i. Also, $R_1 = R - d\cos\theta$ for phase and $R_1 = R$ for magnitude. More generally:

$$\begin{aligned} \text{Phase:} \quad R_i &\approx R - id\cos\theta \\ \text{Magnitude:} \quad R_i &\approx R \end{aligned}$$

Our total field then becomes:

$$\overline{E} = \underbrace{\overline{f}_e(\theta,\phi)\frac{e^{-jkR}}{R}}_{\text{Single Element Radiation}} \underbrace{\sum_{i=0}^{N-1} A_i e^{jkid\cos\theta}}_{f_a(\theta)}$$

So, the total electric field can be written as the product of the single element radiation and an additional factor that takes into account the array. The power density is:

$$\overline{S}(R,\theta,\phi) = \overline{S}_{\text{ff}}(R,\theta,\phi)F_a(\theta)$$

where

$$F_a(\theta) = \left| \sum_{i=0}^{N-1} A_i e^{jkid\cos\theta} \right|^2$$

We call $F_a(\theta)$ the **array factor** for the pattern. It gives the shape of the radiation pattern due to the combination of the multiple elements independent of the shape of the individual element patterns. Often, the array factor dominates the behavior of the total radiation pattern.

1.7.1 Two Element Arrays

There are basic antenna array processes.

- **Analysis**: The individual antenna elements are known and we calculate the pattern that is produced
- **Synthesis**: The antenna pattern is known and we try and determine the individual antenna elements that will produce this pattern

Let's look at two vertical Hertzian dipoles that are:

- separated by $d = \frac{\lambda}{2}$
- have equal amplitudes $a_o = a_1$
- are out of phase by $\pi/2$ $\left(\phi_0 = 1, \phi_1 = \frac{\pi}{2}\right)$

The array factor becomes

$$\begin{aligned} F_a &= \left|1 + e^{-j\frac{\pi}{2}} e^{jkid\cos\theta}\right|^2 && (1.54) \\ &= \left|1 + e^{-j\frac{\pi}{2}} e^{j\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta}\right|^2 && (1.55) \\ &= \left|1 + e^{-j\frac{\pi}{2}} e^{j\pi\cos\theta}\right|^2 && (1.56) \end{aligned}$$

To simplify this we use the following:

$$\begin{aligned} \left|1 + e^{jx}\right| &= \left|2e^{j\frac{x}{2}}\left(e^{-j\frac{x}{2}} + e^{j\frac{x}{2}}\right)\right|^2 && (1.57) \\ &= 4\cos^2\left(\frac{x}{2}\right) && (1.58) \end{aligned}$$

So our original equation becomes

$$F_a = 4\cos^2\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{4}\right) \tag{1.59}$$

The total antenna pattern is then given by

$$S_a = 4S_o \cos^2\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{4}\right) \tag{1.60}$$

This pattern has a maximum when

$$\frac{\pi}{2}\cos\theta - \frac{\pi}{4} = 0 \tag{1.61}$$

or

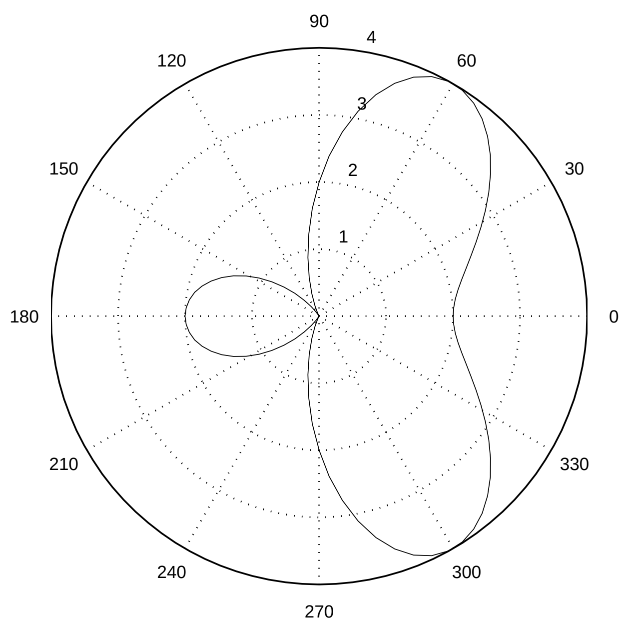
$$\cos\theta = \frac{1}{2} \tag{1.62}$$

$$\theta = \pm 60^o \tag{1.63}$$

and minimums when

$$\frac{\pi}{2}\cos\theta - \frac{\pi}{4} = -\frac{\pi}{2} \tag{1.64}$$

$$\theta = \pm 120^o \tag{1.65}$$



Next let's look at an example of pattern synthesis. In this example we want to use two antennas to produce no radiation in the north/south directions and maximum radiation in the east/west direction.

Again we start with the array factor as given by

$$F_a(\theta) = \left|1 + a_1 e^{j\psi_1} e^{j\frac{2\pi d}{\lambda}\cos\theta}\right|^2 \tag{1.66}$$

We want $F_a = 0$ when $\theta = \pm 90^o$ (North/South direction). Since $\cos 90^o = 0$ the array factor becomes

$$F_a(\theta = 90^o) = \left|1 + a_1 e^{j\psi_1}\right|^2 = 0.$$

This requires $a_1 = a_o = 1$ and $e^{j\psi_1} = -1$ or $\psi_1 = \pi$.

In order to have a maximum at $\theta = 0$ ($\cos(0) = 1$) the array factor becomes

$$F_a(\theta = 90^o) = \left|1 + a_1 e^{j\psi_1} e^{j2\pi d\lambda}\right|^2 = 1,$$

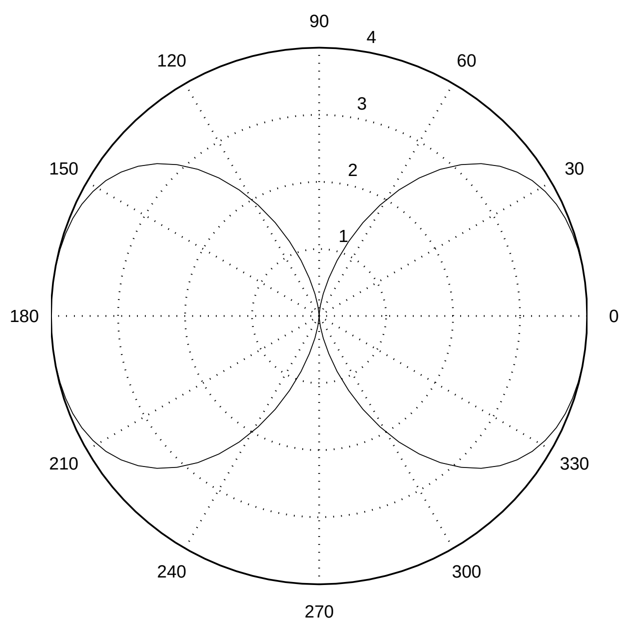
resulting in

$$e^{j\pi} e^{j2\pi d\lambda} = 1 \tag{1.67}$$

$$d = \frac{\lambda}{2} \tag{1.68}$$

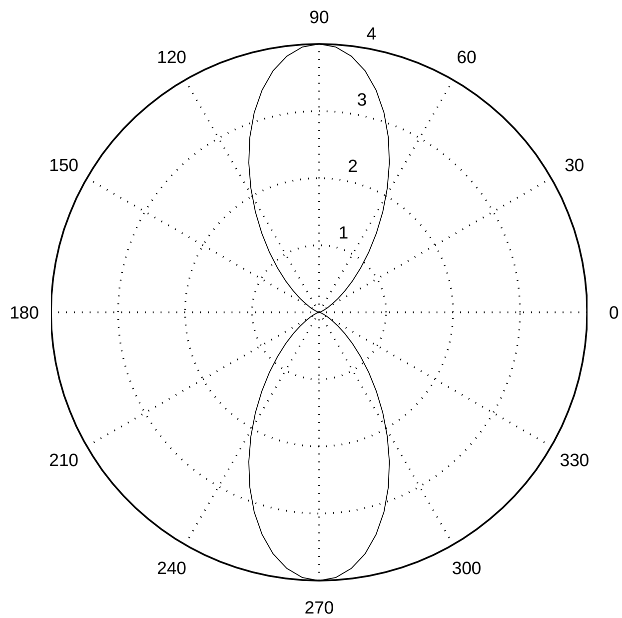
The resulting array factor is

$$\begin{aligned} F_a &= \left|1 - e^{j\pi\cos\theta}\right|^2 && (1.69) \\ &= \left|\left(2j\ e^{j\frac{\pi}{2}}\right)\left(e^{-j\frac{\pi}{2}\cos\theta} - e^{j\frac{\pi}{2}\cos\theta}\right)\right|^2 && (1.70) \\ &= 4\sin^2\left(\frac{\pi}{2}\cos\theta\right) && (1.71) \end{aligned}$$



And if the phase is $\psi_1 = 0$ then the array pattern becomes

$$F_a = 4\cos^2\left(\frac{\pi}{2}\cos\theta\right) \tag{1.72}$$



1.7.2 Uniform Phase Multi-Element Arrays

Let's examine the special case:

$$A_i = e^{ji\psi}$$

We call this a uniformly excited array with a linearly progressive phase.

$$\begin{aligned} F_a(\theta) &= \left| \sum_{i=0}^{N-1} e^{ji(\psi + kd\cos\theta)} \right|^2 \\ &= \left| \sum_{i=0}^{N-1} \left[e^{j(\psi + kd\cos\theta)} \right]^i \right|^2 \end{aligned}$$

This is a geometric series:

$$\sum_{i=0}^{N-1} \gamma^i - \frac{1-\gamma^N}{1-\gamma} \qquad \text{for } |a| \leq 1$$

$$\begin{aligned} F_a(\theta) &= \left| \frac{1 - e^{jN(\psi + kd\cos\theta)}}{1 - e^{j(\psi + kd\cos\theta)}} \right|^2 \\ &= \left| \frac{e^{jN(\psi + kd\cos\theta)/2}}{e^{j(\psi + kd\cos\theta)/2}} \frac{e^{-jN(\psi + kd\cos\theta)/2} - e^{jN(\psi + kd\cos\theta)/2}}{e^{-j(\psi + kd\cos\theta)/2} - e^{j(\psi + kd\cos\theta)/2}} \right|^2 \\ &= \left| \frac{-j2\sin\left[N(\psi + kd\cos\theta)/2\right]}{-j2\sin\left[(\psi + kd\cos\theta)/2\right]} \right|^2 \\ &= \frac{\sin^2\left[N(\psi + kd\cos\theta)/2\right]}{\sin^2\left[(\psi + kd\cos\theta)/2\right]} \end{aligned}$$

Let $u = (\psi + kd\cos\theta)/2$, then

$$F_a(u) = \frac{\sin^2(Nu)}{\sin^2 u}$$

1. At $u = \pm m\pi$:

$$\lim_{u\to\pm m\pi} \frac{\sin Nu}{\sin u} = \lim_{u\to\pm m\pi} N\frac{\cos Nu}{\cos u} = N\frac{\cos(\pm Nm\pi)}{\cos(\pm m\pi)} = N(-1)^{(N-1)m}$$

   So:

$$F_a(\pm m\pi) = N^2\left[(-1)^{(N-1)m}\right]^2 = N^2$$

2. Zeros of $F_a(u)$ occur at $u = m\pi/N$ except for $m = 0$, $\pm N$, $\pm 2N$, etc. (since these points are equivalent to the peaks determined above).

3. Because $\cos\theta = (2u - \psi)/kd$ and $-1 \leq \cos\theta \leq 1$, we have:

$$\begin{aligned} -1 \leq &\ \tfrac{2u-\psi}{kd} \leq 1 \\ -kd \leq &\ 2u - \psi \leq kd \\ \frac{\psi - kd}{2} \leq &\ u \leq \frac{\psi + kd}{2} \end{aligned}$$

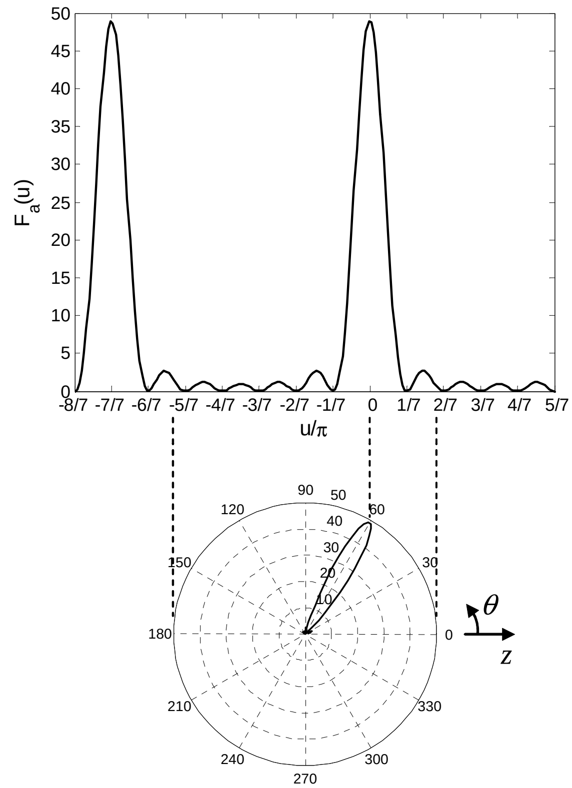
We can use this information to sketch the array factor quite easily. First, we sketch $F_a(u)$. As an example, consider $N = 7$, $\psi = -\pi/2$, and $d = \lambda/2$ ($kd = \pi$). We know that

$$\frac{-\pi/2 - \pi}{2} \leq u \leq \frac{-\pi/2 + \pi}{2} \rightarrow \frac{-3\pi}{4} \leq u \leq \frac{\pi}{4}$$

We call this range of u the **Visible Window**. Again look at the definition

$$u = \frac{\psi}{2} + \frac{kd}{2}\cos\theta$$

This implies that to map u to θ, we draw a circle centered at $\psi/2$ with radius $kd/2$ as shown. We call this technique the visible window technique for plotting array factors. The plot on the right is the array factor in dB.



F_a(u)
u/π
-8/7 -7/7 -6/7 -5/7 -4/7 -3/7 -2/7 -1/7 0 1/7 2/7 3/7 4/7 5/7
θ
z