

Chapter 1

Dynamic Fields

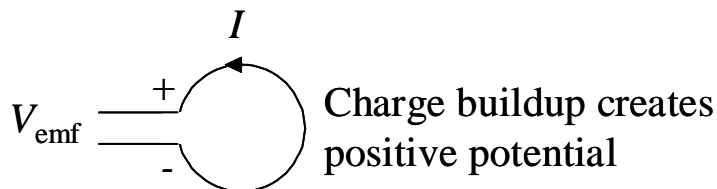
We are now ready to look at the behavior of fields that vary in time. To begin, let's examine a very interesting law: Faraday's Law. We will then look more generally at Maxwell's Equations.

1.1 Faraday's Law

A time-varying magnetic flux through a loop will cause a current to flow in the loop of wire. Faraday's Law describes this effect. To begin, consider a wire loop as shown. A magnetic field passes through the loop (supplied by an external source) such that the flux density vector \vec{B} is normal to the surface defined by the loop. The flux is defined as

$$\phi = \int \int_S \vec{B} \cdot d\vec{s} \quad (1.1)$$

with units of Weber (Wb). When this flux changes in time, a current will flow in the loop. This means that a voltage has been created across the loop terminals called the *electromotive force* (emf).



$$V_{emf} = -\frac{d\phi}{dt} = -\frac{d}{dt} \int \int_S \vec{B} \cdot d\vec{s} \quad (1.2)$$

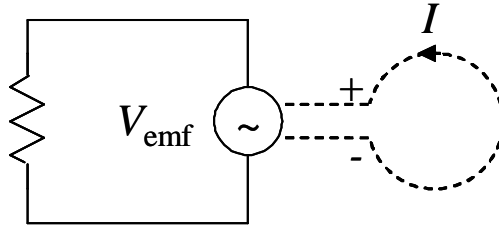
Note that if \vec{B} does not change in time, $V_{emf} = 0$.

The negative sign comes from Lenz's Law, which states that the induced current will oppose the change in flux. To ensure the correct polarity of V_{emf} , we use the right hand rule.

Thumb = direction of $d\vec{s}$

Fingers = direction of + terminal to - terminal

For the above loop, let $d\bar{s}$ be out of the page. If ϕ decreases, V_{emf} is positive. So, V_{emf} represents the voltage available to the load circuit.

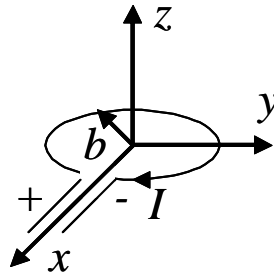


There are two distinct ways to get this emf.

- Transformer emf: A time varying magnetic field linking a stationary loop
- Motional emf: A moving loop with a time varying area (relative to the normal component of \bar{B}) in a static magnetic field

1.1.1 Transformer Action

This is what we've been talking about. The applied \bar{B} through a loop changes in time. The induced potential at the loop terminals is called the transformer emf.



For example, consider the loop shown with

$$\bar{B} = B_0 t \hat{z} \quad (1.3)$$

$$\phi = \int_0^{2\pi} \int_0^b B_0 t r dr d\phi = B_0 t \frac{b^2}{2} 2\pi = B_0 t \pi b^2 \quad (1.4)$$

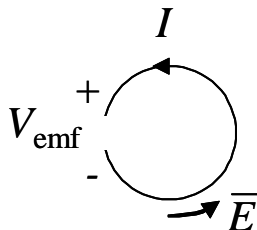
$$V_{emf} = -\frac{d\phi}{dt} = -B_0 \pi b^2 \quad (1.5)$$

\bar{B} is increasing in time, so I is induced as shown to oppose the change. V_{emf} is therefore negative.

Notice also the following. Since $V_{emf} \neq 0$ in this system, $\bar{E} \neq 0$. If we integrate \bar{E} around the loop, we will obtain the voltage V_{emf} . Using the right hand rule for integration direction gives

$$V_{emf} = \oint_C \bar{E} \cdot d\bar{\ell} = -\frac{d}{dt} \int_S \bar{B} \cdot d\bar{s} \quad (1.6)$$

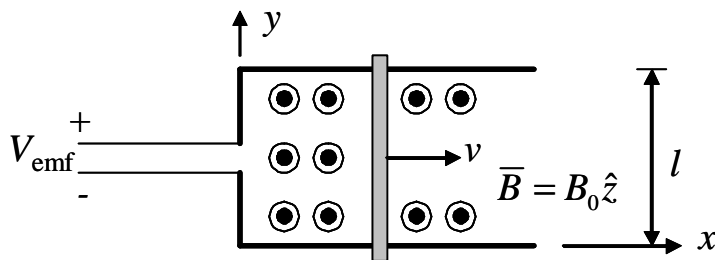
This is **Faraday's Law**. Note that in statics, we had a minus sign to maintain the proper potential reference. However, here the electric field is pushing the charge to make one terminal more positive than the other. Therefore, we do not have the minus sign.



The ideal transformer uses the transformer emf. The time varying voltage creates a time varying magnetic field in the core that has a high magnetic permeability. The time varying magnetic flux induced a current in second winding.

1.1.2 Generator Action

This occurs when the loop is mechanically altered while the flux density remains constant.



$$\phi(t) = \int_0^\ell \int_0^{vt} B_0 dx dy = B_0 \ell vt \quad (1.7)$$

$$V_{\text{emf}} = -\frac{d}{dt}\phi = -B_0 \ell v \quad (1.8)$$

Electromagnetic Generator

A loop of length ℓ and width w is rotating with an angular velocity of ω within a constant magnetic field given by

$$\vec{B} = \hat{z} B_0 \quad (1.9)$$

$$\Phi = \int_S \vec{B} \cdot d\vec{s} \quad (1.10)$$

$$= \int_S \hat{z} B_0 \cdot \hat{n} ds \quad (1.11)$$

$$\hat{n} = \sin\alpha \hat{z} + \cos\alpha \hat{y} \quad (1.12)$$

where

$$\alpha = \omega t + C_o \quad (1.13)$$

$$\Phi = \int_0^\ell \int_0^w B_o \cos\alpha dy dz \quad (1.14)$$

$$= B_o w \ell \cos(\omega t + C_o) \quad (1.15)$$

1.2 Relating Maxwell's Laws in Point and Integral Form

We need to recall once again the vector derivative operations:

Gradient:

$$\nabla\phi(x, y, z) = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z} \quad (1.16)$$

Divergence:

$$\nabla \cdot \bar{F}(x, y, z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (1.17)$$

Curl:

$$\nabla \times \bar{F}(x, y, z) = \hat{x} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad (1.18)$$

We also have two vector identities:

$$\nabla \times \nabla f = 0 \quad (1.19)$$

$$\nabla \cdot (\nabla \times \bar{F}) = 0 \quad (1.20)$$

We also need to consider the integral relations:

Stokes Theorem:

$$\int \int_S \nabla \times \bar{F} \cdot d\bar{s} = \oint_C \bar{F} \cdot d\bar{\ell} \quad (1.21)$$

Divergence Theorem:

$$\int \int \int_V \nabla \cdot \bar{F} dV = \oint_S \bar{F} \cdot d\bar{s} \quad (1.22)$$

Let's use these to derive Maxwell's equations in point form from the equations in integral form:

Faraday's Law:

$$\oint_C \bar{E} \cdot d\bar{\ell} = -\frac{d}{dt} \int \int_S \bar{B} \cdot d\bar{s} \quad (1.23)$$

$$\int \int_S \nabla \times \bar{E} \cdot d\bar{s} = -\frac{d}{dt} \int \int_S \bar{B} \cdot d\bar{s} \quad (1.24)$$

$$\nabla \times \bar{E} = -\frac{\partial}{\partial t} \bar{B} \quad (1.25)$$

Ampere's Law:

$$\oint_C \bar{H} \cdot d\bar{\ell} = \frac{d}{dt} \int \int_S \bar{D} \cdot d\bar{s} + \int \int_S \bar{J} \cdot d\bar{s} \quad (1.26)$$

$$\int \int_S \nabla \times \bar{H} \cdot d\bar{s} = \frac{d}{dt} \int \int_S \bar{D} \cdot d\bar{s} + \int \int_S \bar{J} \cdot d\bar{s} \quad (1.27)$$

$$\nabla \times \bar{H} = \frac{\partial}{\partial t} \bar{D} + \bar{J} \quad (1.28)$$

Gauss' Laws:

$$\oint_S \bar{D} \cdot d\bar{s} = \int \int \int_V \rho_v dV \quad (1.29)$$

$$\int \int \int_V \nabla \cdot \bar{D} dV = \int \int \int_V \rho_v dV \quad (1.30)$$

$$\nabla \cdot \bar{D} = \rho_v \quad (1.31)$$

$$\oint_S \bar{B} \cdot d\bar{s} = 0 \quad (1.32)$$

$$\nabla \cdot \bar{B} = 0 \quad (1.33)$$

1.3 Displacement Current

Consider Ampere's Law:

$$\oint \bar{H} \cdot d\bar{\ell} = \frac{d}{dt} \int \int \bar{D} \cdot d\bar{s} + \int \int \bar{J} \cdot d\bar{s} = \int \int \frac{\partial}{\partial t} \bar{D} \cdot d\bar{s} + I_c \quad (1.34)$$

where I_c represents the conduction current. Notice that the term

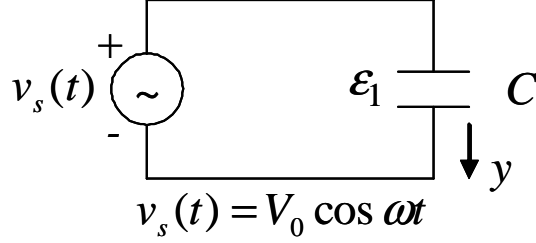
$$I_d = \int \int \frac{\partial}{\partial t} \bar{D} \cdot d\bar{s} \quad (1.35)$$

has units of Amperes. We call this the displacement current.

Consider a parallel plate capacitor as shown. In the wire, $\bar{E} = 0$ and so $\bar{D} = 0$. Therefore, the current is

$$I_w = I_c = \text{conduction current} \quad (1.36)$$

$$= C \frac{dv_s}{dt} = -CV_o \omega \sin \omega t \quad (1.37)$$



In the capacitor, $\bar{J} = 0$ so

$$I_{cap} = I_d = \int \int \frac{\partial}{\partial t} \bar{D} \cdot d\bar{s} \quad (1.38)$$

Since $\bar{E} = (V_o/d) \cos \omega t \hat{y}$, $\bar{D} = (V_o \epsilon_1/d) \cos \omega t \hat{y}$.

$$I_{cap} = - \int \int \frac{V_o \epsilon_1}{d} \omega \sin \omega t \, dx dy = - \underbrace{\frac{\epsilon_1 A}{d}}_C V_o \omega \sin \omega t = -C V_o \omega \sin \omega t \quad (1.39)$$

So: $I_d = I_c$ so that the displacement current allows continuity of current.

1.4 Continuity of Charge

Consider a volume V containing a charge density ρ_v and total charge Q . The only way for Q to change is by charge entering/leaving the surface S bounding V . If I = the net current flowing across S out of V

$$I = -\frac{dQ}{dt} = -\frac{d}{dt} \int \int \int_V \rho_v dV \quad (1.40)$$

But we can also write

$$I = \oint_S \bar{J} \cdot d\bar{s} = -\frac{d}{dt} \int \int \int_V \rho_v dV \quad (1.41)$$

The last term came from Eq. (??). This equation represents the integral form of the continuity of charge. If we now use the Divergence theorem:

$$\oint_S \bar{J} \cdot d\bar{s} = \int \int \int_V \nabla \cdot \bar{J} dV \quad (1.42)$$

$$\nabla \cdot \bar{J} = -\frac{\partial}{\partial t} \rho_v \quad (1.43)$$

This is the continuity of charge in point or differential form.

1.5 Maxwell's Laws in Time-Harmonic Form

To go to sinusoidal steady state, we assume a time variation of $\cos \omega t$. Recall once again that

$$v(t) = \text{Re} \left\{ \tilde{V} e^{j\omega t} \right\} \quad (1.44)$$

Now, we haven't explicitly written it this way yet, but all of the fields are functions of space and time:

$$\bar{E} = \bar{E}(x, y, z, t) = \bar{E}(\bar{R}, t) \quad (1.45)$$

$$\bar{E}(\bar{R}, t) = \text{Re} \left\{ \tilde{E}(\bar{R}) e^{j\omega t} \right\} \quad (1.46)$$

$$\frac{\partial \bar{E}(\bar{R}, t)}{\partial t} = \text{Re} \left\{ \tilde{E}(\bar{R}) j\omega e^{j\omega t} \right\} \quad (1.47)$$

Therefore, Maxwell's equations in time-harmonic (phasor) form are

$$\oint \tilde{E} \cdot d\bar{\ell} = -j\omega \int \int \tilde{B} \cdot d\bar{s} \quad (1.48)$$

$$\oint \tilde{H} \cdot d\bar{\ell} = j\omega \int \int \tilde{D} \cdot d\bar{s} + \int \int \tilde{J} \cdot d\bar{s} \quad (1.49)$$

$$\oint_S \tilde{D} \cdot d\bar{s} = \int \int \int \tilde{\rho}_v dV \quad (1.50)$$

$$\oint_S \tilde{B} \cdot d\bar{s} = 0 \quad (1.51)$$

$$\nabla \times \tilde{E} = -j\omega \tilde{B} \quad (1.52)$$

$$\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J} \quad (1.53)$$

$$\nabla \cdot \tilde{D} = \tilde{\rho}_v \quad (1.54)$$

$$\nabla \cdot \tilde{B} = 0 \quad (1.55)$$

1.6 Boundary Conditions

Our goal is to understand how boundaries in materials impact electric and magnetic fields. Break the electric field and magnetic field into components that are either tangential or normal to the boundary as given by

$$\bar{E}_1 = E_{1t} \hat{t} + E_{1n} \hat{n} \quad (1.56)$$

$$\bar{E}_2 = E_{2t} \hat{t} + E_{2n} \hat{n} \quad (1.57)$$

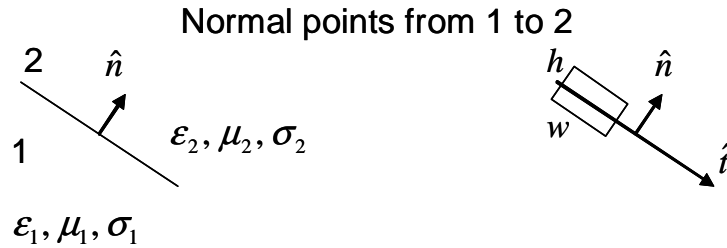
$$\bar{H}_1 = H_{1t} \hat{t} + H_{1n} \hat{n} \quad (1.58)$$

$$\bar{H}_2 = H_{2t} \hat{t} + H_{2n} \hat{n}. \quad (1.59)$$

The goal is to find the relationship between the various components.

We will begin with electric fields. Consider a material boundary as shown with a contour that crosses the boundary. Faraday's law is

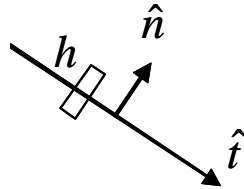
$$\oint_C \bar{E} \cdot d\bar{l} = -\frac{d}{dt} \iint_S \bar{B} \cdot d\bar{s} \quad (1.60)$$



To apply this to the contour, let the contour shrink to zero area so that the right hand side of Faraday's law goes to zero. If we let $w \rightarrow 0$, the left hand side becomes

$$\int_{-h/2}^0 E_{n1} \hat{n} \cdot \hat{n} dl + \int_{-h/2}^0 E_{n1} \hat{n} \cdot (-\hat{n}) dl + \int_0^{h/2} E_{n2} \hat{n} \cdot \hat{n} dl + \int_0^{h/2} E_{n2} \hat{n} \cdot (-\hat{n}) dl = 0 \quad (1.61)$$

In other words, $0 = 0$. This isn't very useful.



If we instead let $h \rightarrow 0$, we get

$$\int_0^w E_{1t} \hat{t} \cdot (-\hat{t}) dl + \int_0^w E_{2t} \hat{t} \cdot \hat{t} dl = 0 \quad (1.62)$$

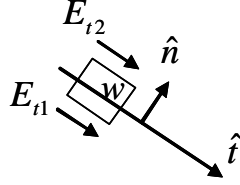
$$\int_0^w (E_{2t} - E_{1t}) dl = 0 \quad (1.63)$$

$$E_{2t} - E_{1t} = 0 \quad (1.64)$$

So, the tangential component of the electric field is **continuous** across the boundary. Note that we can also write

$$\hat{n} \times (\bar{E}_2 - \bar{E}_1) = 0 \quad (1.65)$$

since $\hat{n} \times \bar{E}$ is the tangential electric field.



If we apply the same technique to Ampere's law, we get the same result WITH ONE TWIST.

$$\oint_C \bar{H} \cdot d\bar{\ell} = \frac{d}{dt} \int \int_S \bar{B} \cdot d\bar{s} + \int \int_S \bar{J} \cdot d\bar{s} \quad (1.66)$$

Suppose that \bar{J} represents a surface current that flows in the $\hat{s} = \hat{n} \times \hat{t}$ direction. Note that \hat{s} is tangent to the surface but points normal to the surface of our integration contour.

$$\int_0^w H_{1t} \hat{t} \cdot (-\hat{t}) dl + \int_0^w H_{2t} \hat{t} \cdot \hat{t} dl = \int_0^w J_s dl \quad (1.67)$$

$$\int_0^w (H_{2t} - H_{1t}) dl = \int_0^w J_s dl \quad (1.68)$$

$$H_{2t} - H_{1t} = J_s \quad (1.69)$$

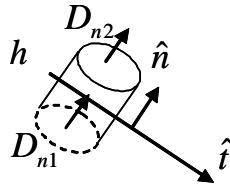
This works out to

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s \quad (1.70)$$

So, tangential \bar{H} can be discontinuous across the interface if a surface current exists.

Let's try Gauss' law. We use a "pillbox" for the integration.

$$\oint_S \bar{D} \cdot d\bar{s} = \int \int \int_V \rho_v dV \quad (1.71)$$



If $h \rightarrow 0$, the right hand side will go to zero unless there is a surface charge density. Then,

$$\int \int_A D_{n2} \hat{n} \cdot \hat{n} ds + \int \int_A D_{n1} \hat{n} \cdot (-\hat{n}) ds = \int \int_A \rho_s ds \quad (1.72)$$

$$D_{n2} - D_{n1} = \rho_s \quad (1.73)$$

$$\hat{n} \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s \quad (1.74)$$

So normal \overline{D} is discontinuous by the surface charge density. For magnetic fields:

$$B_{n2} - B_{n1} = 0 \quad (1.75)$$

$$\hat{n} \cdot (\overline{B}_2 - \overline{B}_1) = 0 \quad (1.76)$$

Normal \overline{B} is continuous. Let's summarize:

Field	Two Dielectrics	Dielectric Conductor
Tangential \overline{E}	Continuous	Continuous: $\hat{n} \times \overline{E} = 0$
Tangential \overline{H}	Continuous	Discontinuous: $\hat{n} \times \overline{H}_1 = 0, \hat{n} \times \overline{H}_2 = \overline{J}_s$
Normal \overline{D}	Continuous	Discontinuous: $\hat{n} \cdot \overline{D}_1 = 0, \hat{n} \cdot \overline{D}_2 = \rho_s$
Normal \overline{B}	Continuous	Continuous: $\hat{n} \cdot \overline{B} = 0$