2.11 Capacitance

One application of Gauss’s law is the analysis of capacitors. The definition of capacitance is

\[ C = \frac{Q}{V} \]  

(2.121)

with units of Farads = Coulombs/Volt, where \( V \) is the potential difference between the conductor with charge \( +Q \) and the conductor with charge \( -Q \).

To compute the capacitance of a given structure, the basic procedure is to assume a charge \( Q \) on the positive conductor, then use Gauss’s law to find the electric field and the voltage between the conductors:

1. Assume a charge \( Q \) on the conductors
2. Find \( \mathbf{D} \) from Gauss’s Law
3. Find \( \mathbf{E} = \mathbf{D}/\epsilon \).
4. Find \( V \) from \( \mathbf{E} \)
5. Find the capacitance from \( C = Q/V \)

2.11.1 Parallel Plate Capacitor

1. Assume the charge on the upper plate is \( Q \). We also assume that the charge will evenly distribute itself over the conducting plates, and neglect fringing fields near the edges of the plates.
2. Use Gauss’s law to find \( \mathbf{D} \). The Gaussian surface is a cube that contains the top plate. Since we are neglecting fringing fields outside the plates, the only contribution to the surface integral is from the part of the Gaussian surface that is between the plates:

\[
\oint \mathbf{D} \cdot d\mathbf{s} = Q \\
\int_0^{L_y} \int_0^{L_x} D_o \, dx \, dy = D_o L_x L_y \\
D_o = \frac{Q}{L_x L_y} = \frac{Q}{A} \\
\mathbf{D} = -\frac{Q}{A} \hat{z}
\]
3. The electric field intensity is
\[ \mathbf{E} = -\frac{Q}{\epsilon A} \hat{z} \]

4. The potential between the two plates is
\[ V = -\int_0^d -\frac{Q}{\epsilon A} \hat{z} \cdot \hat{z} dz = \frac{Qd}{\epsilon A} \]

5. Finally, the capacitance is
\[ C = \frac{Q}{V} = \frac{\epsilon A}{d} \] (2.122)

Question: Why does larger \( \epsilon \) increase the capacitance?

**Differential Forms**

The electric flux density two-form is \( D = -D_o \, dx \wedge dy \), so
\[
\int D = Q \\
\int_0^{L_y} \int_0^{L_x} D_o \, dx \wedge dy = D_o L_x L_y \\
D_o = \frac{Q}{L_x L_y} = \frac{Q}{A} \\
D = -\frac{Q}{A} \, dx \wedge dy
\]

The electric field intensity one-form is
\[ E = \frac{1}{\epsilon} \star D = -\frac{Q}{\epsilon A} \, dz \]

The potential between the two plates is
\[ V = -\int_0^d -\frac{Q}{\epsilon A} \, dz = \frac{Qd}{\epsilon A} \]

and we obtain the same result, \( C = \epsilon A/d \), for the capacitance.

**2.11.2 Energy**

Electromagnetic fields store energy. For electric fields, the energy density is
\[ w_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} \] (2.123)
which has units of Joules/m\(^3\). With differential forms, this becomes a three-form, \((1/2) E \wedge D\). The total energy in a volume \( V \) is
\[ W_e = \frac{1}{2} \int_V E \wedge D \] (2.124)
For the parallel plate capacitor,

\[ W_e = \frac{1}{2} \int_0^d \int_0^{L_y} \int_0^{L_x} \frac{Q^2}{\epsilon A^2} \, dx \wedge dy \wedge dz \]
\[ = \frac{1}{2} Q^2 Ad \]
\[ = \frac{1}{2} C^2 V^2 d \]
\[ = \frac{1}{2} \epsilon A \]
\[ = \frac{1}{2} CV^2 \]  

(2.125)

### 2.11.3 Spherical Shell Capacitor

![Spherical shell capacitor geometry](image)

Figure 2.13: Spherical shell capacitor geometry. Solid lines represent conducting spheres of radius \( a \) and \( b \).
The charge will evenly distribute itself over the conducting spherical shells.

\[ \int_{D} \cdot ds = Q \]  
(2.126)

\[ \int_{0}^{2\pi} \int_{0}^{\pi} D_o R^2 \sin \theta \, d\theta \, d\phi = Q \]  
(2.127)

\[ 4\pi R^2 D_o = Q \]  
(2.128)

\[ D_o = \frac{Q}{4\pi R^2} \]  
(2.129)

Flux density: \( \overrightarrow{D} = \frac{Q}{4\pi R^2} \hat{R} \)  
(2.130)

Field intensity: \( \overrightarrow{E} = \frac{Q}{4\pi \epsilon R^2} \hat{R} \)  
(2.131)

Potential: \( V_{ab} = V_a - V_b = -\int_{b}^{a} \frac{Q}{4\pi \epsilon R^2} \hat{R} \cdot \hat{R} dR \)  
(2.132)

\[ = \frac{Q}{4\pi \epsilon R} \bigg|_{b}^{a} = \frac{Q}{4\pi \epsilon} \left( \frac{1}{a} - \frac{1}{b} \right) \]  
(2.133)

Capacitance: \( C = \frac{Q}{V_{ab}} = \frac{4\pi \epsilon}{\frac{1}{a} - \frac{1}{b}} \)  
(2.134)

### 2.11.4 Electric Field Boundary Conditions

At a boundary between two different dielectrics, the tangential components of \( \overrightarrow{E} \) have to be equal on either side of the interface:

\[ E_{1t} = E_{2t} \]  
(2.135)

If this were not the case, then the closed path integral of \( \overrightarrow{E} \) would not be zero around a small closed path at the boundary. Similarly, the normal components of \( \overrightarrow{D} \) have to be equal on either side of the boundary, or else Gauss’s law would not give zero charge inside a small closed surface at the boundary:

\[ D_{1n} = D_{2n} \]  
(2.136)

If there were charge stored on the boundary, then this changes to \( D_{1n} - D_{2n} = \rho_s \), where \( \rho_s \) is the surface charge density. (A dielectrics will in general have a nonzero surface charge at the boundary due to polarization by an external electric field, but we have already taken this into account when we changed the value of the dielectric constant \( \epsilon \) in Eq. (2.119), so we do not include it on the right-hand side of Gauss’s law.)

\[
\begin{array}{c|c}
\frac{A}{2} & \frac{A}{2} \\
\hline
\epsilon_1 & \epsilon_2 \\
\end{array}
\]

Figure 2.14: Parallel plate capacitor with two different dielectrics.

We can use this to analyze a parallel plate capacitor filled with two different dielectrics, with constants \( \epsilon_1 \) and \( \epsilon_2 \), each occupying half of the area of the plates. Because the electric field intensity \( \overrightarrow{E} \) is tangential to
the interface between the dielectrics, $\mathcal{E}$ must be the same in both dielectrics. So, the flux density is of the form

$$\mathcal{D} = \begin{cases} -\epsilon_1 E_o \hat{z} & \text{region 1} \\ -\epsilon_2 E_o \hat{z} & \text{region 2} \end{cases} \quad (2.137)$$

Applying Gauss’s law,

$$\int_{A/2} \epsilon_1 E_o \hat{z} \cdot dx \, dy \hat{z} + \int_{A/2} \epsilon_2 E_o \hat{z} \cdot dx \, dy \hat{z} = \epsilon_1 E_o A/2 + \epsilon_2 E_o A/2 = Q$$

Solving for $E_o$,

$$E_o = \frac{2Q}{A(\epsilon_1 + \epsilon_2)}$$

The potential is

$$V = -\int \mathcal{E} \cdot d\vec{l}$$

$$= -\int_0^d \frac{-2Q}{A(\epsilon_1 + \epsilon_2)} \hat{z} \cdot d\vec{z}$$

$$= \frac{2Qd}{A(\epsilon_1 + \epsilon_2)}$$

and the capacitance is

$$C = \frac{A(\epsilon_1 + \epsilon_2)}{2d} \quad (2.138)$$

If the dielectric layers were layered one on top of the other instead of side by side, then the flux density would have to be the same in both materials, and the solution would change correspondingly.