Chapter 4

Dynamic Fields

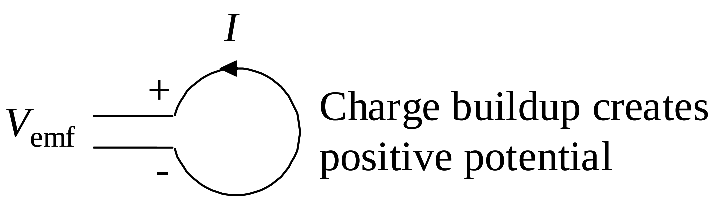
We are now ready to look at the behavior of fields that vary in time. To begin, let's examine a very interesting law: Faraday's Law. We will then look more generally at Maxwell's Equations.

4.1 Faraday's Law

A time-varying magnetic flux through a loop will cause a current to flow in the loop of wire. Faraday's Law describes this effect. To begin, consider a wire loop as shown. A magnetic field supplied by an external source passes through the loop. The flux through the loop is defined as

$$\Lambda = \int_S \overline{B} \cdot d\overline{s} \tag{4.1}$$

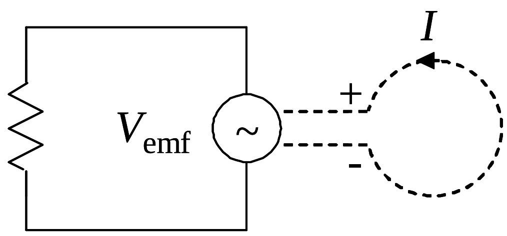
with units of Webers (Wb). When this flux changes in time, a current will flow in the loop. This means that a voltage has been created across the loop terminals called the *electromotive force* (emf):



$$V_{emf} = -\frac{d\Lambda}{dt} = -\frac{d}{dt}\int_S \overline{B} \cdot d\overline{s} \tag{4.2}$$

V_{emf} represents the voltage available to a load circuit attached to the loop. Note that if $\overline{B}$ does not change in time, $V_{emf} = 0$.

The negative sign comes from Lenz's Law, which states that the induced current will oppose the change in flux. To ensure the correct polarity of V_{emf}, we use the right hand rule with the thumb in the direction of $d\overline{s}$, and the fingers give the direction of the $+$ terminal to the $-$ terminal. For the above loop, let $d\overline{s}$ be out of the page. If Λ decreases, V_{emf} is positive.



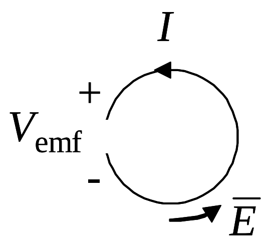
Notice also the following. Since $V_{emf} \neq 0$ in this system, $\overline{E} \neq 0$. If we integrate $\overline{E}$ around the loop, we will obtain the voltage V_{emf}:

$$V_{emf} = \oint_C \overline{E} \cdot d\overline{\ell} = -\frac{d}{dt} \int_S \overline{B} \cdot d\overline{s} \tag{4.3}$$

This is **Faraday's Law** in integral form. The direction of the path integral is given by the right hand rule with respect to $d\overline{s}$. We think of the gap as being small enough that we can consider the integral of $\overline{E}$ to be around a closed path. Since this integral is nonzero, the electric field near a time varying magnetic field is nonconservative.

Before in electrostatics, we had a minus sign in the definition of electric potential because we were finding the potential along a path between two electrodes. Here, the situation is different: the electric field is pushing the charge to make one terminal more positive than the other, so we do not have the minus sign when finding V_{emf}.

For the above example, the right hand side of Eq. (4.3) has a positive value, so the electric field is in the same direction as $d\overline{\ell}$. The direction of $d\overline{\ell}$ is given by the right hand rule with the thumb in the direction of $d\overline{s}$, which is out of the page, so $d\overline{\ell}$ is in the counterclockwise direction. The counterclockwise electric field pushes charge so that the $+$ side becomes positive, making V_{emf} positive. If the flux were increasing, then the electric field would reverse, and the $+$ side would become negatively charged, making V_{emf} negative.

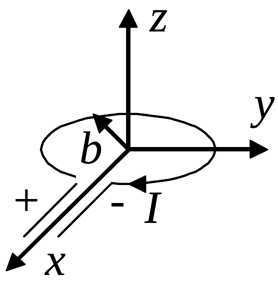


There are two ways to obtain this effect:

- Transformer emf: A time varying magnetic field linking a stationary loop.
- Motional emf: A moving loop with a time varying area (relative to the direction of $\overline{B}$) in a static magnetic field.

4.1.1 Transformer Action

If the applied $\overline{B}$ through a loop changes in time, the induced potential at the loop terminals is called the transformer emf.



For example, consider the loop shown with

$$\overline{B} = B_o t\hat{z} \tag{4.4}$$

$$\Lambda = \int_0^{2\pi}\int_0^b B_o t\, r dr d\phi = B_o t\frac{b^2}{2}2\pi = B_o t\pi b^2 \tag{4.5}$$

$$V_{emf} = -\frac{d\Lambda}{dt} = -B_o\pi b^2 \tag{4.6}$$

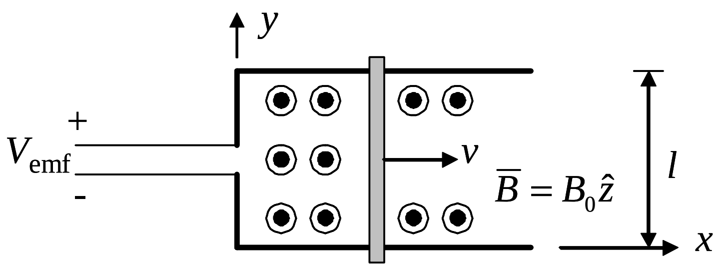
$\overline{B}$ is increasing in time, so I is induced as shown to oppose the change. V_{emf} is therefore negative.

This effect is used in a transformer. A time varying voltage creates a time varying magnetic field in the core that has a high magnetic permeability to increase the flux density. The time varying magnetic flux induces a current in the secondary winding.

4.1.2 Generator Action

This occurs when the loop is mechanically altered while the flux density remains constant.

Sliding Bar



$$\Lambda(t) = \int_0^{\ell}\int_0^{vt} B_o dx dy = B_o\ell vt \tag{4.7}$$

$$V_{emf} = -\frac{d}{dt}\Lambda = -B_o\ell v \tag{4.8}$$

Rotating Generator

A loop of length ℓ and width w is rotating with an angular velocity of ω within a constant magnetic field given by

$$\overline{B} = \hat{z} B_o \tag{4.9}$$

The magnetic flux through the loop is

$$\begin{aligned} \Lambda &= \int_S \overline{B} \cdot d\overline{s} &(4.10)\\ &= \int_S \hat{z} B_o \cdot \hat{n} ds &(4.11) \end{aligned}$$

where $\hat{n} = \cos(\omega t)\,\hat{z} + \sin(\omega t)\,\hat{y}$ and ω is the rotation rate in rad/sec of the loop. So,

$$\begin{aligned} \Lambda &= \int_0^\ell \int_0^w B_o \cos(\omega t)\, ds &(4.12)\\ &= B_o w \ell \cos(\omega t) &(4.13) \end{aligned}$$

The emf is

$$V_{emf} = B_o A \omega \sin(\omega t) \tag{4.14}$$

where $A = w\ell$ is the area of the loop.

4.1.3 Inductor Law

The voltage-current relationship for an inductor is really Faraday's law. The voltage induced across the terminals of a solenoid carrying a current $i(t)$ is

$$\begin{aligned} v(t) &= \oint_C \overline{E} \cdot d\bar{\ell} &(4.15)\\ &= -\frac{d}{dt} \int_S \overline{B} \cdot d\overline{s} &(4.16)\\ &= -\frac{d}{dt} N \int_S -\frac{\mu N i(t)}{\ell}\, ds &(4.17)\\ &= \frac{d}{dt} N \frac{\mu N i(t)}{\ell} \pi a^2 &(4.18)\\ &= \underbrace{\frac{\mu \pi a^2 N^2}{\ell}}_{\text{Inductance } L} \frac{di(t)}{dt} &(4.19)\\ &= L \frac{di(t)}{dt} &(4.20) \end{aligned}$$

The factor of N in Eq. (4.17) is because the surface S is really N disks bounded by each turn of the coil. The extra minus sign in (4.17) arises because the direction of $d\overline{s}$ is opposite to the direction of the magnetic field produced by $i(t)$ if the current flows from the $+$ reference to the $-$ reference.