Chapter 4

Dynamic Fields

We are now ready to look at the behavior of fields that vary in time. To begin, let’s examine a very interesting law: Faraday’s Law. We will then look more generally at Maxwell’s Equations.

4.1 Faraday’s Law

A time-varying magnetic flux through a loop will cause a current to flow in the loop of wire. Faraday’s Law describes this effect. To begin, consider a wire loop as shown. A magnetic field supplied by an external source passes through the loop. The flux through the loop is defined as

\[ \Lambda = \int_S \mathbf{B} \cdot d\mathbf{s} \]  \hspace{1cm} (4.1)

with units of Webers (Wb). When this flux changes in time, a current will flow in the loop. This means that a voltage has been created across the loop terminals called the electromotive force (emf):

\[ V_{\text{emf}} = -\frac{d\Lambda}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \]  \hspace{1cm} (4.2)

\( V_{\text{emf}} \) represents the voltage available to a load circuit attached to the loop. Note that if \( \mathbf{B} \) does not change in time, \( V_{\text{emf}} = 0 \).

The negative sign comes from Lenz’s Law, which states that the induced current will oppose the change in flux. To ensure the correct polarity of \( V_{\text{emf}} \), we use the right hand rule with the thumb in the direction of \( d\mathbf{s} \), and the fingers give the direction of the + terminal to the − terminal. For the above loop, let \( d\mathbf{s} \) be out of the page. If \( \Lambda \) decreases, \( V_{\text{emf}} \) is positive.
Notice also the following. Since $V_{emf} \neq 0$ in this system, $\mathbf{E} \neq 0$. If we integrate $\mathbf{E}$ around the loop, we will obtain the voltage $V_{emf}$:

$$V_{emf} = \oint_{C} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot d\mathbf{s}$$  \hspace{1cm} (4.3)

This is Faraday’s Law in integral form. The direction of the path integral is given by the right hand rule with respect to $d\mathbf{s}$. We think of the gap as being small enough that we can consider the integral of $\mathbf{E}$ to be around a closed path. Since this integral is nonzero, the electric field near a time varying magnetic field is nonconservative.

Before in electrostatics, we had a minus sign in the definition of electric potential because we were finding the potential along a path between two electrodes. Here, the situation is different: the electric field is pushing the charge to make one terminal more positive than the other, so we do not have the minus sign when finding $V_{emf}$.

For the above example, the right hand side of Eq. (4.3) has a positive value, so the electric field is in the same direction as $d\mathbf{l}$. The direction of $d\mathbf{l}$ is given by the right hand rule with the thumb in the direction of $d\mathbf{s}$, which is out of the page, so $d\mathbf{l}$ is in the counterclockwise direction. The counterclockwise electric field pushes charge so that the $+$ side becomes positive, making $V_{emf}$ positive. If the flux were increasing, then the electric field would reverse, and the $+$ side would become negatively charged, making $V_{emf}$ negative.

There are two ways to obtain this effect:

- Transformer emf: A time varying magnetic field linking a stationary loop.
- Motional emf: A moving loop with a time varying area (relative to the direction of $\mathbf{B}$) in a static magnetic field.

### 4.1.1 Transformer Action

If the applied $\mathbf{B}$ through a loop changes in time, the induced potential at the loop terminals is called the transformer emf.
For example, consider the loop shown with
\[
\begin{align*}
\mathcal{B} &= B_o t \hat{z} \quad (4.4) \\
\Lambda &= \int_0^{2\pi} \int_0^b B_o t r dr d\phi = B_o t \frac{b^2}{2} 2\pi = B_o t \pi b^2 \quad (4.5) \\
V_{emf} &= -\frac{d\Lambda}{dt} = -B_o t \pi b^2 \quad (4.6)
\end{align*}
\]

\(\mathcal{B}\) is increasing in time, so \(I\) is induced as shown to oppose the change. \(V_{emf}\) is therefore negative.

This effect is used in a transformer. A time varying voltage creates a time varying magnetic field in the core that has a high magnetic permeability to increase the flux density. The time varying magnetic flux induces a current in the secondary winding.

### 4.1.2 Generator Action

This occurs when the loop is mechanically altered while the flux density remains constant.

**Sliding Bar**

\[
\begin{align*}
\Lambda(t) &= \int_0^t \int_0^{vt} B_o dx dy = B_o vt \quad (4.7) \\
V_{emf} &= -\frac{d\Lambda}{dt} = -B_o vt \quad (4.8)
\end{align*}
\]
Rotating Generator

A loop of length $\ell$ and width $w$ is rotating with an angular velocity of $\omega$ within a constant magnetic field given by

$$\mathbf{B} = \hat{z}B_o$$  \hspace{1cm} (4.9)

The magnetic flux through the loop is

$$\Lambda = \int_S \mathbf{B} \cdot d\mathbf{s}$$  \hspace{1cm} (4.10)

$$= \int_S \hat{z}B_o \cdot \hat{n}ds$$  \hspace{1cm} (4.11)

where $\hat{n} = \cos (\omega t) \hat{z} + \sin (\omega t) \hat{y}$ and $\omega$ is the rotation rate in rad/sec of the loop. So,

$$\Lambda = \int_0^{\ell} \int_0^w B_o \cos (\omega t) \, ds$$  \hspace{1cm} (4.12)

$$= B_o \omega \ell \cos (\omega t)$$  \hspace{1cm} (4.13)

The emf is

$$V_{emf} = B_oA\omega \sin (\omega t)$$  \hspace{1cm} (4.14)

where $A = w\ell$ is the area of the loop.

4.1.3 Inductor Law

The voltage-current relationship for an inductor is really Faraday’s law. The voltage induced across the terminals of a solenoid carrying a current $i(t)$ is

$$v(t) = \oint_C \mathbf{E} \cdot d\ell$$  \hspace{1cm} (4.15)

$$= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$  \hspace{1cm} (4.16)

$$= -\frac{d}{dt} N \int_S -\mu Ni(t) \, ds$$  \hspace{1cm} (4.17)

$$= \frac{d}{dt} N \frac{\mu Ni(t)}{\ell} \pi a^2$$  \hspace{1cm} (4.18)

$$= \frac{\mu \pi a^2 N^2}{\ell} \frac{di(t)}{dt}$$  \hspace{1cm} (4.19)

Inductance $L$

$$= \frac{di(t)}{dt}$$  \hspace{1cm} (4.20)

The factor of $N$ in Eq. (4.17) is because the surface $S$ is really $N$ disks bounded by each turn of the coil. The extra minus sign in (4.17) arises because the direction of $d\mathbf{s}$ is opposite to the direction of the magnetic field produced by $i(t)$ if the current flows from the $+$ reference to the $-$ reference.