2.2 Differential Forms

With vector analysis, there are some operations such as the curl derivative that are difficult to understand physically. We will introduce a notation called the calculus of differential forms that is very similar to vector analysis, but gives us an intuitive way to visualize these operations.

Since it is easy to convert a vector to a differential form and vice versa, we can use either notation to solve a given problem. Mathematically, working with differential forms is very similar to working with vectors, but the pictures that we can draw for fields when we express them as differential forms are quite different and in many cases provide a better way to visualize otherwise complicated phenomena. So, we will use mostly vectors, and convert to differential forms in places where they help in illuminating a difficult point.

2.2.1 What is a Differential Form?

A differential form is a quantity that can be integrated, including the differentials. In the integral below, $3x\,dx$ is a differential form:

$$\int_a^b \underbrace{3x\,dx}_{\text{one-form}}$$

This differential form has degree one because it is integrated over a 1-dimensional region, or path. We call a differential form of degree one a one-form. Differential forms can be added together, with the differentials being linearly independent. The sum of dx and dy, for example, is the one-form $dx + dy$. The sum of $3x\,dx$ and $2\,dx$ is $(3x + 2)\,dx$ since we can combine like differentials.

Two-forms are integrated by double integrals over surfaces. For example, $zx^2dx \wedge dy$ is a two-form. Two-forms under integral signs are written zx^2dxdy, without the wedge. We want to take these differential forms out from under the integral signs so that we can combine them and take derivatives of them, much like we do with vectors. This requires a special rule for combining forms, called the *wedge* or *exterior product* and represented by the symbol $\wedge$, that allows us to think of two-forms as a combination of one-forms.

Table 2.1 shows differential forms of various degrees. Zero-forms and three-forms correspond to scalars or functions. One-forms and two-forms correspond to vectors. It should be obvious why the one-form and vector given in the table go together. The relationship between the two-form and its associated vector will become clear below when we show how to draw pictures of two-forms. Functions are “integrated” by evaluating them at a point, and a point is zero-dimensional, so we can call functions zero-forms.

Degree	Region of Integration	General Expression	Vector/Scalar
zero-form	“Point”	$f(x, y, z)$	$f(x, y, z)$
one-form	Path	$A_xdx + A_ydy + A_zdz$	$A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$
two-form	Surface	$A_xdy \wedge dz + A_ydz \wedge dx + A_zdx \wedge dy$	$A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$
three-form	Volume	$\rho(x, y, z)\,dx \wedge dy \wedge dz$	$\rho(x, y, z)$

Table 2.1: Differential forms.

We will now discuss each type of form separately, and show how to draw pictures for them. Drawing a picture of a vector is easy: a line with an arrow. But there is only one type of picture for vector analysis. With differential forms, there are three pictures. This is one of the things that makes differential forms helpful in electromagnetic theory.

2.2.2 One-forms

A one-form is drawn as surfaces. The one-form dx has surfaces perpendicular to the x-axis spaces one unit apart, as shown in Fig. 2.1(a). $5\,dx$ also has surfaces perpendicular to the x-axis but they are spaced more closely: five per unit distance. This is because of the way one-forms are integrated. If we integrate dx along a path from the point $(.5, 0, 0)$ to the point $(1.5, 0, 0)$, we get one. If we draw the path, it crosses one of the surfaces of dx. With a larger coefficient, the integral is bigger, so the path has to cross more surfaces.

The one-form dy has surfaces perpendicular to the y-axis. The one-form $2\,dz$ has surfaces perpendicular to the z-axis spaced twice as closely as those for dx, as in Fig. 2.1(b).

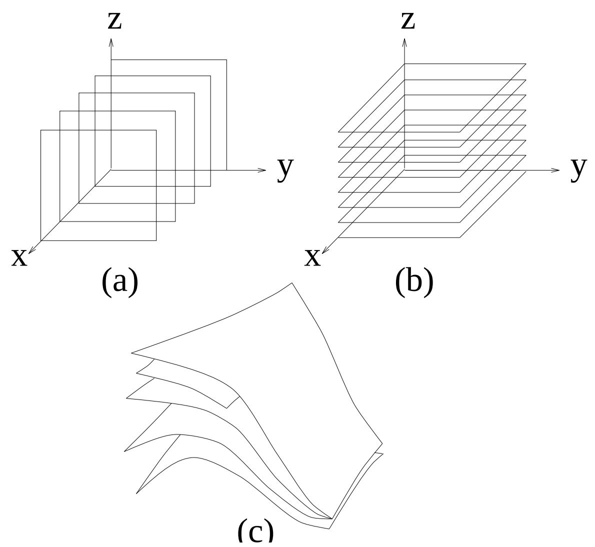

Figure 2.1: (a) The one-form dx. The surfaces of dx are infinite in the z and y directions, and are drawn with edges only for graphical clarity. For the same reason, not all of the surfaces are shown. (b) The one-form $2dz$, with surfaces perpendicular to the z-axis and spaced twice as closely as those of dx. (c) A general one-form, with curved surfaces and surfaces that end or meet each other.

More complicated forms can also be drawn. The one-form $dx + 5dy$ is drawn as slanted surfaces that are perpendicular to the vector $\hat{x} + 5\hat{y}$. The one-form $f dx$ consists of surfaces that are perpendicular to the x-axis but with spacing that gets closer or farther apart depending on the value of the function f. In general, the surfaces of a one-form can twist wildly, end, or meet each other. An example of this is shown in Fig. 2.1(c).

Fig. 2.2 shows how an arbitrary one-form is integrated over a path. The integral of a one-form over a path is the number of surfaces of the one-form pierced by the path. Since we integrate along the path in a particular direction, we have to keep track of the *orientation* of each surface. The orientation of a form is determined by the sign of its coefficients. If a path crosses a surface of the one-form dx in the $+x$ direction, for example, that contributes a positive value to the integral. If the path crosses the surface in the $-x$ direction, that contributes negatively.

In rectangular coordinates, to convert one-forms into vectors and vectors into one-forms, we interchange basis one-forms with basis vectors as below:

$$dx \leftrightarrow \hat{x}, \quad dy \leftrightarrow \hat{y}, \quad dz \leftrightarrow \hat{z}. \tag{2.28}$$

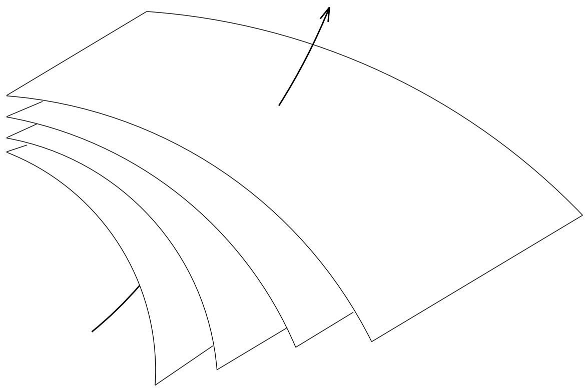

Figure 2.2: Integrating a one-form over a path graphically. Since the path crosses four surfaces, the value of the integral is four.

The vector corresponding to a one-form is sometimes called the one-form's *dual* vector.

2.2.3 Two-forms

A two-form is drawn as two sets of surfaces that intersect to form tubes. To draw $dx \wedge dy$, we superimpose the surfaces of dx and the surfaces of dy as in Fig. 2.3(a). This produces tubes that point in the z direction. This explains the correspondence between the two-form and its dual vector given in Table 2.1.

Graphically, understanding the integral of a two-form over a surface is easy. We just count how many tubes pass through it, as in Fig. 2.3(b). Of course, we have to keep track of the orientation of the two-form (the direction of the tubes) and the orientation of the surface it is being integrated over. A surface is oriented by choosing one of the two normal directions. Tubes crossing in the same direction as the orientation make a positive contribution to the value ot the integral; tubes crossing in the negative directions contribute negatively.

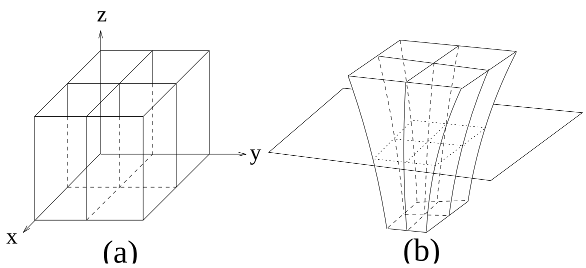

Figure 2.3: (a) The two-form $dx \wedge dy$, with tubes in the z direction. The tubes are infinitely long, but are drawn as finite for clarity. There are also infinitely many tubes in the x and y directions. (b) A two-form is integrated over a surface by counting the number of tubes passing through the surface.

As we noted earlier, between the differentials of a two-form there is a special product, the wedge $\wedge$ or exterior product. Sometimes, the wedge is dropped, so that the two-form $dx\,dy$, for example, is the exterior product of the one-forms dx and dy, or $dx \wedge dy$.

The exterior product is anticommutative, so that switching the order of two differentials changes the sign: $dx \wedge dy = -dy \wedge dx$. One consequence of this property is that $dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$. The anticommutativity of the exterior products allows us to simplify differential forms.

Example 2.1. Exterior product of one-forms.

Let $a = 3dx + dy$ and $b = 2dx + 3dy$. Then

$$\begin{aligned} a \wedge b &= (3dx + dy) \wedge (2dx + 3dy) \\ &= 6dx \wedge dx + 9dx \wedge dy + 2dy \wedge dx + 3dy \wedge dy \\ &= 9dx \wedge dy - 2dx \wedge dy \\ &= 7dx \wedge dy. \end{aligned}$$

This two-form is dual to the cross product $(3\hat{x} + \hat{y}) \times (2\hat{x} + 3\hat{y}) = 7\hat{z}$.

For convenience, we always put differentials of two-forms into the right cyclic orders $dy \wedge dz$, $dz \wedge dx$ and $dx \wedge dy$. Two-forms with differentials in right cyclic order can be converted to vectors by interchanging basis forms and basis vectors as follows:

$$dy \wedge dz \leftrightarrow \hat{x}, \quad dz \wedge dx \leftrightarrow \hat{y}, \quad dx \wedge dy \leftrightarrow \hat{z}. \tag{2.29}$$

There are two types of vectors: those that are dual to one-forms, and those that are dual to two-forms. Usually, vectors dual to two-forms represent flow or flux.

2.2.4 Three-forms

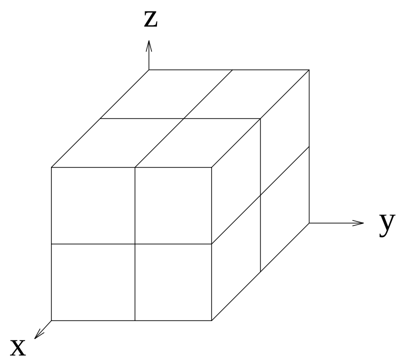

Figure 2.4: The three-form $dx \wedge dy \wedge dz$, with cubes of side equal to one. The cubes fill all space; only a few of them are drawn in the figure.

A three-form has three sets of surfaces that form boxes (Fig. 2.4). The larger the coefficient of the three-form, the smaller and more tightly packed the boxes. The integral of a three-form over a volume is the number of boxes inside the volume, taking into account the sign of the contribution to the integral if the coefficient of the three-form is negative.

We always put the differentials of a three-form in right cyclic order, $dx \wedge dy \wedge dz$. Since any combination of the three differentials dx, dy and dz can be converted to $dx \wedge dy \wedge dz$ using the anticommutativity of the exterior product, any sum of three-forms combines into one term. The coefficient of this term is a scalar. The three-form is dual to this scalar.

A three-form represents a volume density, so the coefficient of a three-form has units of length^{-3}. Some scalars, such as temperature, are not volume densities. The calculus of differential forms lets us keep these two types of quantities separate.

Finally, there are no four-forms, since there are only three differentials, so that any four-form has a repeated differential in each term and so must vanish.

2.3 Integrating Differential Forms

When integrating a vector field over a path or surface, the dot product with a differential vector actually converts the vector field into a differential form. So, it is more natural to integrate a differential form than a vector field.

Consider for example the one-form $\alpha = 2dx + 3xdy$ and a path P which lies along the curve $y = x^2$ from the point $(0, 0)$ to $(1, 1)$. We wish to find

$$\int_P \alpha \tag{2.30}$$

This is done by parameterizing the path P in terms of a new variable t, so that the path becomes $(x = t, y = t^2)$, with t ranging from zero to one. We then substitute these values for x and y into the integral:

$$\begin{aligned}
\int_P \alpha(x,y) &= \int_0^1 \alpha(t,t^2) \\
&= \int_0^1 [2dt + 3td(t^2)] \\
&= \int_0^1 (2 + 6t^2)dt \\
&= 4
\end{aligned}$$

When the symbol d acts on t^2, we use implicit differentiation to obtain $2t\,dt$. Integrating the dual vector $2\hat{x} + 3x\hat{y}$ over the same path gives the same result.

This same approach can be used to evaluate surface integrals of two-forms as well. A parameterization of a surface requires two variables, so that the surface is given by $(x = a(s,t),\ y = b(s,t),\ z = c(s,t))$ where a, b and c are functions of s and t. These functions are substituted into the two-form to be integrated, yielding a two-form with the differentials $ds \wedge dt$, which is then integrated over the appropriate limits in s and t.

2.3.1 Star Operator

The star operator relates zero-forms with three-forms and one-forms with two-forms according to the relationships

$$\star 1 = dx \wedge dy \wedge dz \tag{2.31}$$

and

$$\star dx = dy \wedge dz \tag{2.32}$$
$$\star dy = dz \wedge dx \tag{2.33}$$
$$\star dz = dx \wedge dy \tag{2.34}$$

Also, $\star\star = 1$, so that $\star dy \wedge dz = dx$, for example. Graphically, for a one-form α, the tubes of the two-form $\star\alpha$ are perpendicular to the surfaces of α.

2.3.2 Summary

Differential forms are classified by degree: zero-forms are functions, one-forms are dual to vectors and are drawn as surfaces, two-forms are also dual to vectors but are drawn as tubes, and three-forms are dual to scalars and are drawn as boxes. Differential forms combine using the exterior product to yield differential forms of higher degree. One-forms are integrated over paths, two-forms are integrated over surfaces, and three-forms are integrated over volumes.