8.5 Dipole Antennas

The Hertzian dipole is great because it is easy to formulate the fields for this antenna. However, it is impractical because we cannot effectively radiate power with such an antenna (the radiation resistance is small). The analysis that we used assumes that the current along the dipole is constant. However, for dipoles of a practical length (say a half wavelength), the current is not constant along the dipole, and therefore our analysis is incorrect. We therefore desire to examine this more practical antenna structure.

Before we can do this analysis, however, we need to make some simplifications to our integral for $A$. For realistic currents, we generally cannot perform the integration to compute $A$. However, since we are typically interested in the far-fields, we can make a far-field approximation to the integral.

Let $\hat{R}$ be the unit vector in the direction of the observation vector $\vec{r}$. For a point $\vec{r}$ very far from the source point $\vec{r}'$, we can approximate the value

$$|\vec{r} - \vec{r}'| \approx R - \hat{R} \cdot \vec{r}'$$

(8.17)

So, for the phase term in our Green’s function, we can write

$$e^{-jk|\vec{r} - \vec{r}'|} \approx e^{-jkR} e^{jk\hat{R} \cdot \vec{r}'}$$

(8.18)

For the magnitude, we can simplify this expression even further by neglecting the term $\hat{R} \cdot \vec{r}'$ to write

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{R}$$

(8.19)

We therefore have our far-field approximate form of the magnetic vector potential given as

$$\vec{A}_{\text{ff}}(\vec{r}) = \frac{\mu}{4\pi} \frac{e^{-jkR}}{R} \int \vec{J}(\vec{r}') e^{jk\hat{R} \cdot \vec{r}'}$$

(8.20)

Furthermore, when we take the curl of $\vec{A}_{\text{ff}}$ to obtain the magnetic and electric fields, we neglect any terms that come from this expression that decay faster than $1/R$ (i.e. terms that behave as $1/R^2$, $1/R^3$, etc. This simplification leads to the forms

$$\vec{B}_{\text{ff}} = \nabla \times \vec{A}_{\text{ff}} \approx -jk \hat{R} \times \vec{A}_{\text{ff}}$$

$$\vec{E}_{\text{ff}} = \frac{1}{j\omega} \nabla \times \vec{H}_{\text{ff}} \approx -j\frac{k}{\omega} \hat{R} \times \vec{H}_{\text{ff}} \approx j\omega \hat{R} \times (\hat{R} \times \vec{A}_{\text{ff}})$$
We can now do the integration for a half-wavelength dipole. A reasonable approximation for the current on a dipole is a sinusoid that goes to zero at the ends of the dipole wires, or \( J(r') = \hat{z} I_o \delta(x') \delta(y') \cos(kz') \), \(-\lambda/4 \leq z' \leq \lambda/4\). Then

\[
\mathcal{A}_{ff}(\mathbf{r}) = \frac{\mu_0}{4\pi} e^{-jkR} \int_{-\lambda/4}^{\lambda/4} \hat{z} I_o \cos(kz') e^{jkz' \cos \theta} dz'
\]

\[
= \frac{\mu_0}{8\pi} e^{-jkR} I_o \int_{-\lambda/4}^{\lambda/4} \left[ e^{jkz' \cos \theta + 1} + e^{jkz' \cos \theta - 1} \right] dz'
\]

\[
= (R \cos \theta - \hat{\theta} \sin \theta) \frac{\mu_0}{2k\pi} e^{-jkR} I_o \cos \left[ \frac{\pi}{2} \cos \theta \right] \sin^2 \theta
\]

\[
\mathcal{H}_{ff}(\mathbf{r}) = -\frac{jk}{\mu_0} \hat{R} \times \mathcal{A}_{ff}(\mathbf{r}) = \frac{j I_o}{2\pi} \cos \left[ \frac{\pi}{2} \cos \theta \right]
\]

\[
E_{ff} = j\omega R (\hat{R} \times \mathcal{A}_{ff}) = \frac{j}{2\pi} I_o e^{-jkR} \cos \left[ \frac{\pi}{2} \cos \theta \right] \sin \theta
\]

The time-average Poynting vector is:

\[
S_{av,R} = \left| \frac{E_{ff}}{2\eta_0} \right|^2 = \eta_0 |I_o|^2 \left\{ \frac{\cos \left[ \frac{\pi}{2} \cos \theta \right]}{\sin \theta} \right\}^2
\]

This Poynting vector is maximum at \( \theta = \pi/2 \) with the maximum being

\[
S_{max} = \frac{\eta_0 |I_o|^2}{8(\pi R)^2}
\]

Therefore, the radiation pattern is:

\[
F(\theta) = \left\{ \frac{\cos \left[ \frac{\pi}{2} \cos \theta \right]}{\sin \theta} \right\}^2
\]

With this radiation pattern, we can determine:

Radiated Power: \( P_{rad} = 36.6 |I_o|^2 \)

Directivity: \( D = 1.64 \)

Radiation Resistance: \( R_{rad} = 73 \Omega \)
8.6 Receiving

Antennas are also used for capturing energy from an incident wave and converting it into power.

The power collected by the receiving antenna depends on the power density of the incident wave and the effective collecting area of the antenna as given by

\[ P_{rec} = S_i A_e, \quad (8.21) \]

where \( S_i \) is the power density of the incident wave, \( P_{rec} \) is the power collected by the receiver, and \( A_e \) is the effective collecting area of the receiving antenna.

The basic derivation process is:

1. Calculate the amount of power collected by a Hertzian dipole.
2. Relate this to an effective area the collected power.
3. Generalize to an arbitrary antenna by relating the effective area to the directivity.

These following derivation assume that (1) the antenna is impedance matched to the transmission line and (2) the antenna loss is low \( (R_{loss} \ll R_{rad}) \).

The first step is to calculate the collected power for a given incident power density. The load is matched to the antenna using \( Z_L = Z_{in}^* \). The load current is thus given by

\[ I_L = \frac{V_{oc}}{Z_{in} + Z_L} = \frac{V_{oc}}{2R_{rad}} \]

The received power is

\[ P_{rec} = \frac{1}{2} |I_L|^2 R_{rad} \]

\[ = \frac{1}{2} \frac{|V_{oc}|^2}{(2R_{rad})^2} R_{rad} = \frac{|V_{oc}|^2}{8R_{rad}} \quad (8.22) \]

The incident power density is related to the incident electric field as given by

\[ S_i = \frac{|E_i|^2}{2\eta_0} \approx \frac{|E_i|^2}{240\pi} \]

The effective area of the antenna is

\[ A_e = \frac{P_{rec}}{S_i} = \frac{|V_{oc}|^2}{8R_{rad}} \frac{240\pi}{|E_i|^2} = \frac{|V_{oc}|^2}{|E_i|^2} \frac{30\pi}{R_{rad}} \quad (8.23) \]

For a Hertzian dipole the field is constant across the antenna, resulting in

\[ V_{oc} = E_i \ell \]

We found before that

\[ R_{rad} = 80\pi^2 \left( \frac{\ell}{\lambda} \right)^2 \]
The effective area can then be calculated to be

\[ A_e = \frac{\left| E_i \ell \right|^2}{E_i^2} \frac{30\pi}{80\pi^2} \left( \frac{\lambda}{\ell} \right)^2 \]

\[ = \frac{3\lambda^2}{8\pi} \]  

(8.24)

Relating this to the gain of a Hertzian dipole results in

\[ A_e = \frac{\lambda^2 G}{4\pi} \]  

(8.25)

Although we derived this for a Hertzian dipole, this same expression can be used to define the effective area of any antenna.

Now we want to couple the transmitting and receiving antennas together to get a complete link. We start with calculating the power density at the location of the receiver produced by the transmitting antenna, which can be found from the definition of gain:

\[ G_t = \frac{\text{Power Density}}{\text{Power density of an isotropic radiator}} \]

\[ = \frac{S_i}{P_t} \frac{4\pi R^2}{4\pi R^2} \]

\[ = S_i \left( \frac{4\pi R^2}{P_t} \right) \]

From this, we find that the incident power density is

\[ S_i = G_t \left( \frac{P_t}{4\pi R^2} \right) \]  

(8.26)

Now we determine the power collected by the receiving antenna using

\[ P_{\text{rec}} = S_i A_r. \]  

(8.27)

We relate the effective area to the antenna gain to get

\[ P_{\text{rec}} = S_i G_r \frac{\lambda^2}{4\pi} \]  

(8.28)

Finally, we plug in the expression for the incident power density to give

\[ P_{\text{rec}} = \left( G_t \frac{P_t}{4\pi R^2} \right) G_r \left( \frac{\lambda^2}{4\pi} \right) \]  

(8.29)

This results in the Friis transmission formula,

\[ \frac{P_{\text{rec}}}{P_t} = G_t G_r \left( \frac{\lambda}{4\pi R} \right)^2 \]  

(8.30)

The last factor on the right is sometimes called the **free space path loss**.
Example

A satellite to ground link is established for satellite TV with the following system parameters:

- \( P_t = 100 \text{ W} \)
- \( L = 40,000 \text{ km} \)
- Minimum detectable power \( P_{\text{rec}} = 1 \text{ pW} \)
- The antennas are essentially lossless

The effective area of a dish antenna is approximately equal to the area of the dish. In order to keep the price down the transmitting antenna is chosen to be 4 times larger than the antenna on the ground, so that \( A_t = 4 A_r \). What is the diameter of the receiving antenna?

\[
\frac{P_{\text{rec}}}{P_t} = G_t G_r \left( \frac{\lambda}{4\pi R} \right)^2 \quad (8.31)
\]
\[
\frac{10^{-12}}{100} = A_r A_t \left( \frac{4\pi}{\lambda^2} \right)^2 \left( \frac{\lambda}{4\pi R} \right)^2 \quad (8.32)
\]
\[
10^{-14} = A_r A_t \left( \frac{1}{\lambda R} \right)^2 \quad (8.33)
\]

Since we chose \( A_t = 4 A_r \), we get

\[
10^{-14} = 4 A_r^2 \left( \frac{1}{\lambda R} \right)^2 \quad (8.34)
\]
\[
10^{-14} = 4 A_t^2 \left( \frac{1}{\lambda R} \right)^2 \quad (8.35)
\]

\[
A_r = \sqrt{\frac{10^{-14}}{4} \times 4 \times 10^7 \lambda}
= 2 \lambda \quad (8.36)
\]

If the frequency is C-band \((f=4 \text{ GHz})\) then the antenna diameter is \( d = 0.4 \text{ m} \).