3.2 Magnetic Vector Potential

In electrostatics, we had the notion of a potential. This concept is useful, since it is sometimes more convenient to compute the potential and then compute the electric field using \( E = -\nabla V \). It would be convenient to also define a magnetic potential to assist in the computation of magnetic fields. But because even magnetostatic fields are nonconservative, we cannot define a scalar magnetic potential. But we can define a vector quantity which mathematically acts like a potential, in that we can obtain the magnetic field from it by taking a derivative. We call this quantity the magnetic vector potential and denote it as \( \vec{A} \).

In constructing an equation for \( \vec{A} \), we use Gauss’s Law of magnetics that \( \nabla \cdot \vec{B} = 0 \). We also take advantage of a vector identity that for any vector \( \vec{A} \),

\[
\nabla \cdot (\nabla \times \vec{A}) = 0
\]

Therefore, we will use

\[
\vec{B} = \nabla \times \vec{A}
\]

so that we are guaranteed to satisfy Gauss’s Law. Since \( \vec{B} \) has units Wb/m\(^2\), \( \vec{A} \) has units of Wb/m.

Since \( \vec{B} = \mu \vec{H} \), Ampere’s Law gives

\[
\nabla \times \vec{B} = \mu \vec{J}
\]

\[
\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}
\]

Now, we invoke a second vector identity that states

\[
\nabla^2 \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})
\]

where \( \nabla^2 \vec{A} \) is the vector Laplacian operation. We can therefore write

\[
\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}
\]

Now, when we define \( \vec{B} = \nabla \times \vec{A} \), we have not uniquely specified the vector \( \vec{A} \). In order to uniquely specify a vector, we must specify its curl and its divergence. Therefore, we are still free to choose the divergence of \( \vec{A} \) any way we like. It is clearly convenient to specify

\[
\nabla \cdot \vec{A} = 0
\]

Our equation therefore becomes

\[
\nabla^2 \vec{A} = -\mu \vec{J}
\]

So, we now have a differential equation for \( \vec{A} \) much like we had an equation for \( V \). A solution to this differential equation (which we will not prove) is

\[
\vec{A} = \frac{\mu}{4\pi} \int_V \frac{\vec{J}}{R'} dV'
\]

where \( R \) is the distance from the integration point to the point where the field is observed, or \( R' = |\vec{R} - \vec{R}_i| \). Here, \( \vec{R} \) is the point where \( \vec{A} \) is evaluated. \( \vec{R}_i \) is the point of integration. If we have a current distribution, we can compute \( \vec{A} \) using this integral and then compute \( \vec{B} = \nabla \times \vec{A} \).
3.3 Magnetic Properties of Materials

We need to briefly discuss the magnetic permeability $\mu$ much like we discussed the permittivity for electrostatics. Using a classical description of matter, all atoms have electrons which orbit the nucleus. This orbiting charge represents a current loop which creates a magnetic moment. In most materials, called diamagnetic materials, these atoms are randomly aligned so that there is no net magnetic effect.

The electrons also spin which creates another contribution to the magnetic moment. In atoms with even numbers of electrons, there are always two spins that are equal but opposite, resulting in zero net magnetic moment from spin. For an odd number of electrons, there is a net magnetic effect from the single unpaired electron.

When a material is exposed to a magnetic field $\overline{H}$, we can express the magnetic flux density as

$$\overline{B} = \mu_0 \overline{H} + \mu_0 \overline{M} = \mu_0 (\overline{H} + \overline{M})$$  \hfill (3.46)

where $\overline{M}$ is called the magnetization vector of a material. This vector represents the vector sum of the magnetic dipole moments of the atoms. Physically, the magnetic field is aligning the atomic magnetic dipoles. The degree to which these dipoles can be aligned is represented by the magnitude of $\overline{M}$. Much like we did in electrostatics, we define a magnetic susceptibility $\chi_m$ and write

$$\overline{M} = \chi_m \overline{H}$$  \hfill (3.47)

$$\overline{B} = \mu_0 (\overline{H} + \chi_m \overline{H}) = \mu_0 \left(1 + \chi_m \right) \overline{H} = \mu_r \overline{H}$$  \hfill (3.48)

The units of $\mu$ are H/m. For most materials, $\chi_m$ is so small that we can write $\mu_r = 1$. Ferromagnetic materials, however, which are susceptible to magnetic alignment, can have high values of $\mu_r$. For example, pure iron has $\mu_r = 2 \times 10^5$.

Types of magnetic materials:

1. Diamagnetic: $\chi_m < 0$ ($\chi_m \sim -10^{-5}$, $\mu_r \sim 1$)
2. Paramagnetic: $\chi_m > 0$ ($\chi_m \sim 10^{-5}$, $\mu_r \sim 1$)
3. Ferromagnetic: $|\chi_m| \gg 1$

3.4 Inductance

The physical behavior of currents and magnetic fields is interesting. For example, we know that currents create magnetic fields. However, a static magnetic field (created by a magnet) which cuts through a loop of wire will not create a current unless the loop or magnet are moving.

We will see that time-varying magnetics fields can induce currents. For example, if we have two separate loops, and drive a time-varying current through one, we will observe a current in the other. This magnetic linking is mutual inductance. Similarly, in a solenoid, the behavior of the current in one loop is influenced by time-varying current in another loop. This is self inductance, since the actual current flowing through both loops is the same.
Despite the fact that we need time-varying currents/fields to have this coupling, we can compute the strength of the coupling using static analysis. The strength of the coupling is the inductance of the system:

\[
\text{Inductance: } L = \frac{\Lambda}{I}
\]  

(3.49)

where

\[
\Lambda = \text{net magnetic flux linkage} = \text{net flux linking current}
\]

\[
I = \text{current producing flux linkage}
\]

### 3.4.1 Steps for Computing Inductance

1. Assume a current and a form for \( \overline{H} \)
2. Calculate \( \overline{H} \) using Ampere’s Law
3. Calculate \( \overline{B} = \mu \overline{H} \)
4. Calculate \( \Lambda = \int \overline{B} \cdot d\overline{s} \)
5. Calculate \( L = \Lambda / I \)

### 3.4.2 Solenoid

1.-3. The magnetic flux density for a current \( I \) was computed previously in Eq. (3.33):

\[
\overline{B} = \mu \frac{NI}{\ell} \hat{z}
\]  

(3.50)

4. For a single loop, the flux passing through the loop is

\[
\Lambda_1 = \int_0^{2\pi} \int_0^a \overline{B} \cdot d\overline{s}
\]  

(3.51)

\[
= \int_0^{2\pi} \int_0^a \mu \frac{NI}{\ell} \hat{z} \cdot \hat{z} rd\phi
\]  

(3.52)

\[
= \mu 2\pi \frac{a^2}{2} \frac{NI}{\ell}
\]  

(3.53)

\[
= \mu \pi a^2 \frac{NI}{\ell}
\]  

(3.54)

For \( N \) loops, the flux linkage is

\[
\Lambda = N\Lambda_1 = \mu \pi a^2 \frac{N^2 I}{\ell}
\]  

(3.55)

5. The inductance is

\[
L = \frac{\Lambda}{I} = \frac{\mu \pi a^2 N^2}{\ell}
\]  

(3.56)

You may wonder why we use \( \Lambda = N\Lambda_1 \). If we pass a different time-varying current \( i(t) \) through the solenoid, then the voltage across one loop would be \( v(t) = L_1 di/dt = (\Lambda_1/I)di/dt \). Across all the loops in the inductor, the voltages add in series, so that \( v(t) = (N\Lambda_1/I)di(t)/dt \).
3.4.3 Toroid

Based on our prior work, steps 1-3 yield

\[
\bar{B} = \mu \frac{NI}{2\pi r} \hat{\phi}
\]  
(3.57)

The integral in step 4 is relatively difficult to perform, so let’s make an approximation. If the radius \( a \) of the core of the toroid is small compared to the radius of the toroid (i.e., \( b \gg a \)), then

\[
\bar{B} \approx \mu \frac{NI}{2\pi b} \hat{\phi}
\]  
(3.58)

To obtain the net flux, we integrate over the area of the toroid cross-section:

\[
\Lambda_1 = \int \int \mu \frac{NI}{2\pi b} \hat{\phi} dr dz 
\]  
(3.59)

\[
\Lambda_1 = \mu \frac{NI}{2\pi b} \frac{a^2}{2} = \mu \frac{NIa^2}{2b}
\]  
(3.60)

\[\Lambda = NA_1 = \mu \frac{N^2Ia^2}{2b}
\]  
(3.61)

The inductance is therefore:

\[
L = \frac{\Lambda}{I} = \mu \frac{N^2a^2}{2b}
\]  
(3.62)

3.4.4 Coax

We have already computed

\[
\bar{B} = \frac{\mu I}{2\pi r} \hat{\phi}
\]  
(3.63)

The flux linking the two conductors is now the flux passing through the area between the conductors:

\[
\Lambda = \int_a^b \int_0^\ell \frac{\mu I}{2\pi r} \hat{\phi} dz dr 
\]  
(3.64)

\[
\Lambda = \frac{\mu I \ell}{2\pi} \ln \left( \frac{b}{a} \right)
\]  
(3.65)

The inductance is

\[
L = \frac{\mu I \ell}{2\pi} \ln \left( \frac{b}{a} \right)
\]  
(3.66)

\[
\frac{L}{\ell} = \frac{\mu}{2\pi} \ln \left( \frac{b}{a} \right)
\]  
(3.67)
3.4.5 Parallel Flat Conductors

\[ \mathbf{H} \approx H_0 \hat{y} \]

\[ \int_0^w H_0 \hat{y} \cdot \hat{y} dy = H_0 w = I \quad (3.69) \]

\[ \mathbf{H} = \frac{I}{w} \hat{y} \quad (3.70) \]

\[ \mathbf{B} = \frac{\mu I}{w} \hat{y} \quad (3.71) \]

\[ \Lambda = \int_0^d \int_0^\ell \frac{\mu I}{w} dz dx = \frac{\mu I}{w} \ell d \quad (3.72) \]

\[ \frac{L}{\ell} = \frac{\mu}{w} \quad (3.73) \]

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