5.4 Lossy Media

Let us now consider the wave equation without assuming $\sigma = 0$:

$$\nabla^2 \overline{E} - \gamma^2 \overline{E} = 0 \tag{5.96}$$

$$\gamma^2 = -\omega^2 \mu \epsilon_c = -\omega^2 \mu(\epsilon - j\sigma/\omega) = -\omega^2 \mu(\epsilon' - j\epsilon'') \tag{5.97}$$

The complex propagation constant γ has real and imaginary parts,

$$\gamma = j\omega\sqrt{\mu\epsilon_c} = \alpha + j\beta \tag{5.98}$$

which can be found from

$$\gamma^2 = (\alpha + j\beta)^2 = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon'' \tag{5.99}$$

$$\alpha^2 - \beta^2 + j2\alpha\beta = -\omega^2\mu\epsilon' + j\omega^2\mu\epsilon'' \tag{5.100}$$

Equating real and imaginary parts and solving, we get:

$$\alpha = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]\right\}^{1/2} \text{ Np/m} \tag{5.101}$$

$$\beta = \omega\left\{\frac{\mu\epsilon'}{2}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]\right\}^{1/2} \text{ rad/m} \tag{5.102}$$

5.4.1 Plane Waves

Now, if we simplify the wave equation for a uniform plane wave just like we did in the lossless case, we obtain

$$\frac{d^2E_x}{dz^2} - \gamma^2 E_x = 0 \tag{5.103}$$

The solution to this differential equation is

$$E_x(z) = E_{xo}^+ e^{-\gamma z} + E_{xo}^- e^{\gamma z} \tag{5.104}$$

$$= E_{xo}^+ e^{-\alpha z} e^{-j\beta z} + E_{xo}^- e^{\alpha z} e^{j\beta z} \tag{5.105}$$

So, the wave decays as it propagates. Note that this also means we take $\alpha > 0$, $\beta > 0$ when we take the square root of γ^2.

To determine $\overline{H}$, we use Faraday's law, $\nabla \times \overline{E} = -j\omega\mu\overline{H}$, so that

$$\overline{H} = \frac{1}{\eta_c}\hat{k} \times \overline{E} \tag{5.106}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \tag{5.107}$$

where η_c is the intrinsic impedance of the lossy medium. Since η_c is a complex number, $\overline{E}$ and $\overline{H}$ are no longer in phase.

5.4.2 Skin Depth

For a $+z$-traveling wave, the magnitude of the electric field is

$$|E_x(z)| = |E^+_{xo}E^{-\alpha z}e^{-j\beta z}| = |E^+_{xo}|e^{-\alpha z} \tag{5.108}$$

The propagation distance required to attenuate the wave by a factor of e^{-1} is called the *skin depth* δ_s:

$$|E_x(z=\delta_s)| = |E^+_{xo}|e^{-1} \rightarrow \delta_s = \frac{1}{\alpha} \tag{5.109}$$

Extremes:	Perfect Conductor	$\sigma = \infty$	$\alpha = \infty$	$\delta_s = 0$
	Dielectric	$\sigma = 0$	$\alpha = 0$	$\delta_s = \infty$

When an AC current flows in a conductor, since $\overline{E}$ decays rapidly, the current $\overline{J} = \sigma\overline{E}$ is concentrated near the conductor surface. In a perfect conductor, the current becomes a surface current density. This is called the skin effect.

5.4.3 Loss Tangent

The loss tangent is simply a commonly-used parameter to describe the loss of a medium. It is defined as:

$$\text{Loss Tangent} = \tan\delta = \frac{\epsilon''}{\epsilon'} \tag{5.110}$$

Often, materials are specified by ϵ' and $\tan\delta$ at a certain frequency:

Polystyrene Foam:	$\epsilon' = 1.03\epsilon_0$	$\tan\delta = 0.3 \times 10^{-4}$	$f = 3$GHz
Fresh Snow:	$\epsilon' = 1.20\epsilon_0$	$\tan\delta = 3 \times 10^{-4}$	$f = 3$GHz
Round Steak:	$\epsilon' = 40\epsilon_0$	$\tan\delta = 0.3$	$f = 3$GHz

Let's put the round steak in the microwave oven (not my favorite way to prepare steak). The complex permittivity is

$$\epsilon_c = \epsilon' - j\epsilon'' = \epsilon'\left(1 - j\frac{\epsilon''}{\epsilon'}\right) = \epsilon'\left(1 - j\tan\delta\right) \tag{5.111}$$

$$= 40\left(1 - j0.3\right)\epsilon_0 \tag{5.112}$$

$$\gamma = j\omega\sqrt{\mu_0\epsilon_c} = j\omega\sqrt{\mu_0\epsilon_0}\sqrt{40(1-j0.3)} = j\frac{2\pi}{\lambda_0}\sqrt{40(1-j0.3)} \tag{5.113}$$

At f = 3GHz, $\lambda_0 = 10$cm = 0.1m:

$$\gamma = \alpha + j\beta = 59 + j402\text{m}^{-1} \tag{5.114}$$

$$\delta_s = \frac{1}{\alpha} = 0.017\text{m} = 1.7\text{cm} \tag{5.115}$$

So, the microwave oven heats the surface more rapidly that it heats the center (contrary to popular belief). However, it is true that a microwave immediately starts heating the center (not all heat arrives at the center through heat conduction). For polystyrene foam:

$$\epsilon_c = 1.03\left(1 - j0.3 \times 10^{-4}\right)\epsilon_0 \tag{5.116}$$

$$\gamma = 9.6 \times 10^{-4} + j63.8\text{m}^{-1} \tag{5.117}$$

Since α is so small, very little wave attenuation (and therefore heating) occurs. This is why you can reheat your meat in a styrofoam box in the microwave without the box getting hot.

5.5 Parameter Simplifications

We now want to look at approximations for α and β in Eqs. (5.101) and (5.102), in order to get a more qualitative understanding of how the various material properties (σ, f, ϵ_r) parameters affect plane wave propagation in different types of materials. We will look at two cases: (1) if $\epsilon'' \ll \epsilon'$ the material is a low loss medium, and (2) if $\epsilon'' \gg \epsilon'$ the material is a good conductor.

Low loss. For a low-loss dielectric, the expression for γ can be put in the form

$$\gamma = j\omega\sqrt{\mu\epsilon'}\ \left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} \tag{5.118}$$

The second square root can be approximated using the first two terms of the binomial expansion $\sqrt{1+\Delta} \simeq 1 + \Delta/2$. This results in

$$\gamma \simeq j\omega\sqrt{\mu\epsilon'}\ \left(1 - j\frac{\epsilon''}{2\epsilon'}\right) \tag{5.119}$$

The real and imaginary part are

$$\alpha \simeq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}} \tag{5.120}$$

$$\beta \simeq \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu\epsilon}. \tag{5.121}$$

This expression shows that β is the same as in the lossless case, so the plane propagation behavior for a low-loss medium is the same with the addition of a decay term.

The intrinsic impedance is also approximated using the binomial expansion as given by

$$\eta \simeq \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right) \tag{5.122}$$

$$\simeq \sqrt{\frac{\mu}{\epsilon}}, \tag{5.123}$$

which is the same as it was for the lossless case.

Good conductor. For the case $\epsilon'' \gg \epsilon'$,

$$\gamma = \sqrt{-\omega^2\mu(\epsilon' - j\epsilon'')} \tag{5.124}$$

$$\simeq \sqrt{j\omega^2\mu\epsilon''} \tag{5.125}$$

We now substitute $\epsilon'' = \sigma/\omega$ and $\sqrt{j} = (1+j)/\sqrt{2}$ to get

$$\gamma = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j) \tag{5.126}$$

resulting in

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}. \tag{5.127}$$

In this case the propagation and decay constants are equal and change with frequency. The approximation for the intrinsic impedance follows a similar derivation, resulting in

$$\eta_c = \sqrt{j\frac{\mu}{\epsilon''}} = (1+j)\sqrt{\frac{\pi f \mu}{\sigma}}. \tag{5.128}$$

With a complex η the electric and magnetic fields are no longer in phase.

Is it valid to assume that dielectrics are low-loss and metals are good conductors?

$$\begin{array}{cc} \text{Dielectric} & \text{Conductor} \\ 1 \gg \frac{\epsilon''}{\epsilon'} & 1 \ll \frac{\epsilon''}{\epsilon'} \\ \frac{1}{100} > \frac{\epsilon''}{\epsilon'} & 100 < \frac{\epsilon''}{\epsilon'} \\ \frac{1}{100} > \frac{\sigma}{\omega\epsilon_r\epsilon_o} & 100 < \frac{\sigma}{\omega\epsilon_r\epsilon_o} \\ \omega > \frac{100\sigma}{\epsilon_r\epsilon_o} & \omega < \frac{\sigma}{100\epsilon_r\epsilon_o} \\ \omega > \frac{100\times10^{-12}}{(4)(8.854\times10^{-12})} & \omega < \frac{10^6}{(100)(8.854\times10^{-12})} \\ \omega > 2.8 \text{ rad/s} & \omega < 10^{12} \text{ rad/s} \end{array} \tag{5.129}$$

This shows that the approximations are very valid for dielectrics and conductors over a wide frequency range.

Sea Water Example

Let's look at plane wave propagation through sea water. The material parameters are

$\epsilon_r = 72 - 80$ (We will use $\epsilon_r = 80$)

$\sigma = 4$

What is the range for the good conductor approximation?

$$\frac{\epsilon''}{\epsilon'} > 100 \tag{5.130}$$

$$\frac{\sigma}{\omega\epsilon_r\epsilon} > 100 \tag{5.131}$$

$$\omega < 56 \text{ MHz} \tag{5.132}$$

What is the range for the low-loss dielectric approximation?

$$\frac{\epsilon''}{\epsilon'} < \frac{1}{100} \tag{5.133}$$

$$\frac{\sigma}{\omega\epsilon_r\epsilon} < \frac{1}{100} \tag{5.134}$$

$$\omega > 565 \text{ GHz} \tag{5.135}$$

At 1 KHz, the decay constant is

$$\alpha\,(1\text{ kHz}) = \sqrt{\pi\cdot10^3\cdot4\cdot4\pi\times10^{-7}} = 0.126 \text{ np/m} \tag{5.136}$$

and the skin depth is about 8 meters.

At microwave frequencies, neither approximation is valid, so we have to return to the original expression for γ to find the skin depth. So, for 1 GHz,

$$\begin{aligned}\gamma = j\omega\sqrt{\mu_0\epsilon_c} &= j2\pi 10^9\sqrt{4\pi\times 10^{-7}8.854\times 10^{-12}(80 - j4/(2\pi\times 10^9))}\\ &\simeq 78 \text{ Np/m} + j203 \text{ rad/m}\end{aligned}$$

The skin depth is about 1.3 cm.

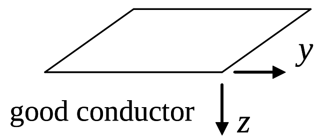
5.5.1 Current Flow in Good Conductors (Skin Effect)

If we have a DC current, the current will be uniformly distributed across the conductor cross section. However, in the AC case, the current is concentrated near the conductor surface.

Consider a semi-infinite slab of conducting material. A plane wave exists in the medium whose fields just below the top surface are expressed as:

$$\overline{E}(z = 0^+) = \hat{x}E_0 \quad (5.137)$$

$$\overline{H}(z = 0^+) = \hat{y}\frac{E_0}{\eta_c} \quad (5.138)$$



So, the plane waves are:

$$\overline{E}(z) = \hat{x}E_0 e^{-\alpha z}e^{-j\beta z} \quad (5.139)$$

$$\overline{H}(z) = \hat{y}\frac{E_0}{\eta_c}e^{-\alpha z}e^{-j\beta z} \quad (5.140)$$

The current flows in the x direction, and the current density is $\overline{J} = \sigma\overline{E} = \hat{x}\sigma E_0 e^{-\alpha z}e^{-j\beta z} = \hat{x}J_0 e^{-\alpha z}e^{-j\beta z}$. If $\epsilon'' \gg \epsilon'$ (good conductor), then $\alpha = \beta = 1/\delta_s$. So,

$$\overline{J} = \hat{x}J_0 e^{-(1+j)z/\delta_s} \quad (5.141)$$

Now, we explore the amount of current flowing through the region $0 \le y \le w$ and $0 \le z < \infty$.

$$\begin{aligned}I = \int_0^w\int_0^\infty J_0 e^{-(1+j)z/\delta_s}dzdy &= -J_0 w\frac{\delta_s}{1+j}\left[e^{-(1+j)\infty/\delta_s} - e^0\right]\\ &= J_0 w\frac{\delta_s}{1+j} \quad (5.142)\end{aligned}$$

Let's suppose we integrate in z only over the following ranges:

Integral in z over	**Error in calculating I is**
$0 \le z \le 3\delta_s$	5%
$0 \le z \le 5\delta_s$	1%

Therefore, we can treat the conductor as infinitely thick as long as the thickness is larger than about $5\delta_s$. The basic principle is that the majority of the current flows within a few skin depths of the surface. For example, for copper we have:

$$\begin{aligned}\sigma_c &= 5.8\times 10^7 \text{ S/m}\\ \delta_s &= \frac{1}{\sqrt{\pi f\mu\sigma_c}} = 2.1\mu\text{m at } f = 1 \text{ GHz}\end{aligned}$$

So, 99% of the current flows within 10μm of the surface.

Resistance

Remember that the impedance is the voltage divided by the total current. The voltage along a path of length l in the x direction is

$$V = -\int \overline{E} \cdot dl \tag{5.143}$$

$$= El \tag{5.144}$$

The impedance is then given by

$$Z = \frac{V}{I} \tag{5.145}$$

$$= (E_o\, l) \left(\frac{1+j}{\sigma E_o\, w\, \delta_s} \right) \tag{5.146}$$

$$= \underbrace{\frac{1+j}{\sigma \delta_s}}_{Z_s} \frac{l}{w} \tag{5.147}$$

where the quantity Z_s is called surface impedance. The surface resistance is

$$R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

As an example, let's look at the resistance per unit length of a coaxial transmission line. The width of the inner conductor is $w_{inner} = 2\pi a$ and of the outer conductor is $w_{outer} = 2\pi b$. The resulting resistance per unit length is then

$$R' = \sqrt{\frac{\pi f \mu}{\sigma}} \left(\frac{1}{w_{inner}} + \frac{1}{w_{outer}} \right) \tag{5.148}$$

$$= \sqrt{\frac{\pi f \mu}{\sigma}} \left(\frac{1}{2\pi} \right) \left(\frac{1}{a} + \frac{1}{b} \right) \tag{5.149}$$

This result takes into account the skin effect.