4.4 Relating Maxwell’s Laws in Point and Integral Form

We need to recall once again the vector derivative operations:

Gradient:
\[ \nabla f(x, y, z) = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z} \quad (4.47) \]

Curl:
\[ \nabla \times \mathbf{F}(x, y, z) = \hat{x} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{y} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{z} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \quad (4.48) \]

Divergence:
\[ \nabla \cdot \mathbf{F}(x, y, z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (4.49) \]

We also have two vector identities:
\[ \nabla \times \nabla f = 0 \quad (4.50) \]
\[ \nabla \cdot (\nabla \times \mathbf{F}) = 0 \quad (4.51) \]

We also need to consider the integral relations:

Stokes’ Theorem:
\[ \int_S \nabla \times \mathbf{F} \cdot d\mathbf{s} = \oint_C \mathbf{F} \cdot d\mathbf{\ell} \quad (4.52) \]

Divergence Theorem:
\[ \int_V \nabla \cdot \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot d\mathbf{s} \quad (4.53) \]

These are generalizations of the fundamental theorem of calculus to two and three dimensions. The fundamental theorem of calculus relates the integral of a derivative of a function over an interval to the value of the function at the endpoints or boundary of the interval. Similarly, Stokes’ theorem relates the integral of the derivative of a vector field over a surface to the integral of the vector field over the boundary of the surface.

Let’s use these theorems to derive Maxwell’s equations in point form from the equations in integral form:

Faraday’s Law:
\[ \oint_C \mathbf{E} \cdot d\mathbf{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (4.54) \]
\[ \int_S \nabla \times \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (4.55) \]
\[ \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \quad (4.56) \]
Ampere’s Law:

\[
\oint_C \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{D} \cdot dS + \int_S \mathbf{J} \cdot dS \tag{4.57}
\]

\[
\int_S \nabla \times \mathbf{H} \cdot dS = \frac{d}{dt} \int_S \mathbf{D} \cdot dS + \int_S \mathbf{J} \cdot dS \tag{4.58}
\]

\[\nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D} + \mathbf{J} \tag{4.59}\]

Gauss’ Laws:

\[
\oint_S \mathbf{D} \cdot dS = \int_V \rho_v dV \tag{4.60}
\]

\[
\int_V \nabla \cdot \mathbf{D} dV = \int_V \rho_v dV \tag{4.61}
\]

\[\nabla \cdot \mathbf{D} = \rho_v \tag{4.62}\]

\[
\oint_S \mathbf{B} \cdot dS = 0 \tag{4.63}
\]

\[\nabla \cdot \mathbf{B} = 0 \tag{4.64}\]

4.5 Continuity of Charge

Consider a volume \( V \) containing a charge density \( \rho_v \) and total charge \( Q \). The only way for \( Q \) to change is by charge entering/leaving the surface \( S \) bounding \( V \). If \( I \) is the net current flowing across \( S \) out of \( V \), then

\[I = -\frac{dQ}{dt} = -\frac{d}{dt} \int_V \rho_v dV \tag{4.65}\]

But we can also write

\[I = \oint_S \mathbf{J} \cdot dS = -\frac{d}{dt} \int_V \rho_v dV \tag{4.66}\]

where the last term comes from Eq. (4.65). This equation represents the integral form of the continuity of charge. If we now use the Divergence theorem, we obtain

\[
\oint_S \mathbf{J} \cdot dS = \int_V \nabla \cdot \mathbf{J} dV \tag{4.67}
\]

\[\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} \rho_v \tag{4.68}\]

This is the continuity of charge in point form.
4.6 Differential Forms

Let’s redo this using differential forms notation. The exterior derivative operator is

\[ d = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge \]  

(4.69)

This is analogous to the \( \nabla \) operator in vector notation, except that we combine it with differential forms using the wedge operation \( \wedge \) instead of the dot or cross products:

\[
\text{Gradient} = d(\text{zero-form}) = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge f
= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \text{one-form}
\]

\[
\text{Curl} = d(\text{one-form}) = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge (F_x dx + F_y dy + F_z dz)
= \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) dz \wedge dx + \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dx \wedge dy = \text{two-form}
\]

\[
\text{Divergence} = d(\text{two-form}) = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge (F_x dy \wedge dz + F_y dz \wedge dx + F_z dx \wedge dy)
= \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dx \wedge dy \wedge dz = \text{three-form}
\]

The exterior derivative of a \( p \)-form is a \( (p + 1) \)-form. The exterior derivative of a three-form is a four-form, which must have a repeated differential and so is zero.

If we apply the exterior derivative to any differential form twice, we get zero \( (dd = 0) \).

The exterior derivative also satisfies a product rule analogous to the product rule for the partial derivative,
\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \]  
where \( p \) is the degree of \( \alpha \).

The generalized Stokes theorem is

\[ \int_M d\omega = \int_{bd M} \omega \]  
(4.70)

where \( \omega \) is a \( p \)-form and \( M \) is a \( (p + 1) \)-dimensional region of integration with boundary \( bd M \). If \( p = 0 \), this is the fundamental theorem of calculus. If \( p = 1 \), this is analogous to the vector Stokes theorem. For \( p = 2 \), this is the divergence theorem.
Using differential forms, Maxwell’s equations in point form are

\[
\begin{align*}
\text{d}E &= -\frac{\partial B}{\partial t} \\
\text{d}H &= \frac{\partial D}{\partial t} + J \\
\text{d}D &= \rho \\
\text{d}B &= 0
\end{align*}
\]

(4.71)  (4.72)  (4.73)  (4.74)

The continuity equation is \(\text{d}J = -\partial \rho/\partial t\).

Graphically,

The exterior derivative of a function is a one-form with surfaces that are level sets for the function.

The exterior derivative of a one-form is a two-form with tubes along the edges of the surfaces of the one-form wherever they end.

The exterior derivative of a two-form is a three-form with boxes wherever tubes of the two-form end or begin.

Figure 4.3: (a) Exterior derivative of a zero-form is a one-form drawn as the level sets of the function. (b) Exterior derivative of a one-form is a two-form with tubes where the one-form surfaces end. (c) Exterior derivative of a two-form is a three-form with boxes where the two-form tubes end or begin.