

5.6 Electromagnetic Power Density

5.6.1 Poynting Vector

Consider Maxwell's equations in the time domain modified as follows:

$$\overline{\mathcal{H}} \cdot \nabla \times \overline{\mathcal{E}} = -\overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} \quad (5.150)$$

$$\overline{\mathcal{E}} \cdot \nabla \times \overline{\mathcal{H}} = \overline{\mathcal{E}} \cdot \frac{\partial \overline{\mathcal{D}}}{\partial t} + \overline{\mathcal{E}} \cdot \overline{\mathcal{J}} \quad (5.151)$$

The divergence of a cross product can be expanded using the identity $\nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) = \overline{\mathcal{H}} \cdot \nabla \times \overline{\mathcal{E}} - \overline{\mathcal{E}} \cdot \nabla \times \overline{\mathcal{H}}$, which is analogous to the product rule for the scalar derivative. Therefore,

$$\nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) = -\overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t} - \overline{\mathcal{E}} \cdot \frac{\partial \overline{\mathcal{D}}}{\partial t} - \overline{\mathcal{E}} \cdot \overline{\mathcal{J}} \quad (5.152)$$

To further simplify this expression, we use

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \mu \|\overline{\mathcal{H}}\|^2 \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} \mu \overline{\mathcal{H}} \cdot \overline{\mathcal{H}} \right) = \frac{1}{2} \mu \left(\frac{\partial \overline{\mathcal{H}}}{\partial t} \cdot \overline{\mathcal{H}} + \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{H}}}{\partial t} \right) = \overline{\mathcal{H}} \cdot \frac{\partial (\mu \overline{\mathcal{H}})}{\partial t} = \overline{\mathcal{H}} \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t}$$

Therefore,

$$\nabla \cdot (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) + \frac{\partial}{\partial t} \left(\frac{1}{2} \mu \|\overline{\mathcal{H}}\|^2 \right) + \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon \|\overline{\mathcal{E}}\|^2 \right) + \sigma \overline{\mathcal{E}} \cdot \overline{\mathcal{E}} = 0$$

Let's integrate over a volume V and apply the divergence theorem to the first term:

$$\oint_S (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{s} + \frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \mu \|\overline{\mathcal{H}}\|^2 + \frac{1}{2} \epsilon \|\overline{\mathcal{E}}\|^2 \right] dV + \int_V \sigma \|\overline{\mathcal{E}}\|^2 dV = 0 \quad (5.153)$$

This is **Poynting's Theorem** and represents a power balance or conservation of energy for electromagnetic fields. The units of each term (after integration) is Watts. We identify each term as:

$$\begin{aligned} \oint_S (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{s} &= \text{total power leaving the volume (through the surface } S) \\ \frac{1}{2} \mu \|\overline{\mathcal{H}}\|^2 &= \text{stored magnetic energy density inside } V \\ \frac{1}{2} \epsilon \|\overline{\mathcal{E}}\|^2 &= \text{stored electric energy density inside } V \\ \frac{\partial}{\partial t} \int_V \left[\frac{1}{2} \mu \|\overline{\mathcal{H}}\|^2 + \frac{1}{2} \epsilon \|\overline{\mathcal{E}}\|^2 \right] dV &= \text{rate of } \mathbf{increase} \text{ of stored energy inside } V \\ \int_V \sigma \|\overline{\mathcal{E}}\|^2 dV &= \text{power lost to heat inside } V \end{aligned} \quad (5.154)$$

Therefore, the theorem states that: *The power leaving the volume + the rate of increase in the stored energy + the power going into heat = 0.* Note that $\overline{\mathcal{E}} \times \overline{\mathcal{H}}$ has units of W/m^2 . It represents the density of power carried by electromagnetic waves across the surface S . We call it the **Poynting Vector**:

$$\overline{\mathcal{S}} = \overline{\mathcal{E}} \times \overline{\mathcal{H}} \quad (5.155)$$

Note that this is analogous to instantaneous power $p(t) = v(t)i(t)$, except that $\overline{\mathcal{S}}$ has units W/m^2 .

In the phasor domain, the average power delivered to a load in a circuit is

$$P = \frac{1}{2} \operatorname{Re}\{\tilde{V}\tilde{I}^*\}$$

Similarly, the time-average Poynting vector indicates the average real power density of a time-harmonic wave:

$$\bar{S}_{av} = \frac{1}{2} \operatorname{Re}\{\bar{E} \times \bar{H}^*\} \quad (5.156)$$

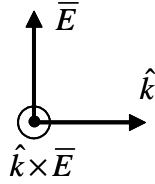
where $\bar{S} = \bar{E} \times \bar{H}^*$ is the complex Poynting vector.

So, there are three power quantities associated with electromagnetic fields:

1. Instantaneous Poynting vector (time-domain): $\bar{S} = \bar{\mathcal{E}} \times \bar{\mathcal{H}}$.
2. Time-average Poynting vector (time-harmonic fields): $\bar{S}_{av} = \frac{1}{T} \int_0^T \bar{S} dt, T = 2\pi/\omega$.
3. Complex Poynting vector (phasor domain): $\bar{S} = \bar{E} \times \bar{H}^*$

The time-average power can be obtained from the complex Poynting vector using $\bar{S}_{av} = \frac{1}{2} \operatorname{Re}\{\bar{E} \times \bar{H}^*\}$.

5.6.2 Poynting Vector for Plane Waves



We know that $\bar{H}(z) = \hat{k} \times \bar{E}(z)/\eta_c$. Therefore

$$\bar{S} = \bar{E} \times \bar{H}^* = \frac{1}{\eta_c^*} \bar{E} \times (\hat{k} \times \bar{E})^* = \frac{1}{\eta_c^*} |\bar{E}|^2 \hat{k}$$

The time-average power flux is

$$\bar{S}_{av} = \frac{1}{2} \operatorname{Re}\{\bar{E} \times \bar{H}^*\} = \frac{|\bar{E}|^2}{2} \operatorname{Re}\left\{\frac{1}{\eta_c^*}\right\} \hat{k}$$

For lossless media,

$$\bar{S}_{av} = \frac{|\bar{E}|^2}{2\eta} \hat{k} \quad (5.157)$$

So, the power flow is in the direction of propagation of the wave (\hat{k}) and the coefficient is analogous to V^2/R .

Solar Illumination

$|\overline{S}_{av}| = 1 \text{ kW/m}^2$ at the Earth's surface due to sun radiation.

$R_e = 6380 \text{ km} = \text{earth radius}$

$R_s = 1.5 \times 10^8 \text{ km} = \text{radius of earth's orbit around sun}$

1. Find the total power radiated by the sun

$$P_{sun} = S_{av}(4\pi R_s^2) = 2.8 \times 10^{26} W$$

2. Find the total power intercepted by the earth. The earth's cross sectional area is $A_e = \pi R_e^2$.

$$P_{earth} = S_{av}\pi R_e^2 = 1.28 \times 10^{17} W$$

3. Find the electric field strength at the earth (assuming single frequency)

$$S_{av} = \frac{|E_0|^2}{2\eta_0} \rightarrow |E_0| = \sqrt{2\eta_0 S_{av}} = 870 \text{ V/m}$$

Wireless Signals

How does the power from a wireless communications system antenna decrease with range?

DC Current

Consider a battery attached by wires to a resistor. How does energy move from the battery to the resistor? Does power flow inside the wires? The surprising answer is no.

We will apply Poynting's theorem to this system for a closed surface that surrounds the battery but does not include the resistor. Since the currents are DC, the stored energy term in Eq. (5.153) is zero because the time derivative is zero. If we assume an ideal battery, then the power loss term is also zero because $\sigma = 0$. We have to add another term to account for the power supplied by the battery, which is like the $\overline{\mathcal{E}} \cdot \overline{\mathcal{J}}$ term in Eq. (5.152) except that the current is an impressed current instead of an induced current. So, Poynting's theorem becomes

$$\oint_S (\overline{\mathcal{E}} \times \overline{\mathcal{H}}) \cdot d\overline{s} + \int_V \overline{\mathcal{E}} \cdot \overline{\mathcal{J}}_{battery} dV = 0 \quad (5.158)$$

The term on the right represents the electrical power supplied by the battery, which according to Poynting's theorem must be balanced by an accompanying outflow of power across the closed surface S , represented by the term containing $\overline{\mathcal{E}} \times \overline{\mathcal{H}}$. If the wires are ideal, then $\overline{\mathcal{E}} = 0$ inside the wires, so the only place the power flow $\overline{\mathcal{E}} \times \overline{\mathcal{H}}$ is zero is inside the wires!

Differential forms provide a nice picture for what actually happens. The potential produced by the battery causes surfaces of E that extend away from the battery to infinity. The current flowing down the wires produces surfaces of H that extend from one wire to the other wire. The surfaces of E and the surfaces of H intersect to form tubes of the Poynting two-form $E \wedge H$ that extend from the battery to the resistor. The

power supplied by the battery and dissipated in the resistor flows along these tubes. The tubes do follow the wires, but they are outside of the wires.

If the battery were replaced by a charged capacitor, then the currents and voltages in the system would no longer be DC. Some of the power would flow along tubes of $E \wedge H$ to infinity, representing power radiated away from the circuit as waves, and the rest would be dissipated in the resistor. The larger the resistor, the slower the capacitor discharge, and the less power is radiated away. In the limit as the resistance becomes zero, no power is absorbed, and all the power radiates away as waves.