1.3 Sinusoidal Steady State

Electromagnetics applications can be divided into two broad classes:

- **Time-domain**: Excitation is not sinusoidal (pulsed, broadband, etc.)
  - Ultrawideband communications
  - Pulsed radar
  - Digital signals
- **Time-harmonic**: Excitation is sinusoidal
  - Narrowband communication schemes - amplitude modulation (AM), frequency modulation (FM), phase shift keying (PSK), etc.
  - Continuous wave radar
  - Optical communications

Time-harmonic systems are fundamental to applications of electrical engineering. The concept of sharing a communication channel by using carrier sinusoids with different frequencies together with receivers tuned to discriminate among the carriers dates back to the earliest days of radio communications.

1.3.1 Phasor Notation

In analyzing time-harmonic systems, we assume that the signal of interest is narrowband enough that it can be approximated as a sinusoid. This approximation works very well for many important applications. Due to capacitance, inductance, and propagation delays in a system, the phase of a signal depends on where the signal is measured. For this reason, it takes two parameters to characterize the signal at any point in the system:

\[
v(x, y, z, t) = v_0(x, y, z) \cos \left( \omega t + \phi(x, y, z) \right)
\]

where \( \omega \) is the time frequency of the signal in radians per meter. It is inconvenient to have the phase \( \phi(x, y, z) \) inside the argument of the cosine function. Dealing with time-harmonic signals is much easier if we express these two degrees of freedom in a more symmetric way, as the real and imaginary parts of a complex number, which we call a phasor. The definition of a phasor voltage \( \tilde{V} \) is

\[
v(x, y, z, t) = \Re \left\{ \tilde{V}(x, y, z) e^{j\omega t} \right\}
\]

At this point, we will drop the \( x \) and \( y \) dependence, and assume that for a transmission line the voltage only depends on time and the position \( z \) along the line.

How does the complex number \( \tilde{V} \) relate to the real voltage \( V \)? By placing the complex number \( \tilde{V} \) in polar form, we can express the voltage as

\[
v(z, t) = \Re \left\{ |\tilde{V}(z)| e^{j\angle\tilde{V}(z)} e^{j\omega t} \right\}
\]

\[
= |\tilde{V}(z)| \Re \left\{ e^{j(\omega t + \angle\tilde{V}(z))} \right\}
\]

\[
= |\tilde{V}(z)| \cos (\omega t + \angle\tilde{V}(z))
\]
By comparing this to Eq. (1.58), we can see that the magnitude of the voltage is equal to the magnitude of the phasor and the phase shift of the voltage relative to $\omega t$ is equal to the phase of $\tilde{V}$. Keep in mind that there is no such thing as a complex voltage. The real and imaginary parts of the phasor voltage $\tilde{V}$ simply offer a convenient tool for keeping track of the magnitude and phase in Eq. (1.58) at different locations in a circuit or system.

Another simplification that results from the use of phasor notation is that time derivatives become multiplication by $j\omega$, through the use of the identity

$$\frac{\partial v(z,t)}{\partial t} = \text{Re}\left\{ j\omega \tilde{V}(z)e^{j\omega t} \right\}$$

(1.63)

The current-voltage relationship for a capacitor, for example, is

$$i(t) = C \frac{dv(t)}{dt}$$

(1.64)

In the phasor domain, this becomes

$$\tilde{I} = j\omega C \tilde{V}$$

(1.65)

The inverse relationship is

$$\tilde{V} = \tilde{I} \frac{1}{j\omega C}$$

(1.66)

Impedance

The beauty of this result is that the capacitor current-voltage relationship now has the form of Ohm’s law, but with an imaginary value in place of resistance. So, we can handle resistors, capacitors, and inductors without having to solve differential equations by using phasor notation.

### 1.3.2 Time-harmonic Wave Equation

By substituting Eqs. (1.59) and (1.63) into the wave equation (1.14), we obtain the time-harmonic wave equation,

$$\frac{d^2 \tilde{V}}{dz^2} = \left[ RG + j\omega(RC + LG) - \omega^2 LC \right] \tilde{V}$$

(1.67)

$$= (R + j\omega L)(G + j\omega C)\tilde{V}$$

(1.68)

The constant on the right-hand side is the square of the complex propagation constant, with the symbol $\gamma$, so that

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$

(1.69)

The solution to the ordinary differential equation (1.68) has the form

$$\tilde{V}(z) = Ae^{mz} + Be^{-mz}$$

(1.70)

Substituting this expression into Eq. (1.68) leads to

$$m^2 \tilde{V} = \gamma^2 \tilde{V}$$

(1.71)

so that the general solution can be expressed as

$$\tilde{V}(z) = Ae^{\gamma z} + Be^{-\gamma z}$$

(1.72)
The constants $A$ and $B$ are determined by the excitation and boundary conditions on the transmission line.

In general, $\gamma$ is complex, and can be expressed in terms of its real and imaginary parts as

$$\gamma = \alpha + j\beta$$

Using this in the general solution leads to

$$\tilde{V}(z) = Ae^{\alpha z}e^{j\beta z} + Be^{-\alpha z}e^{-j\beta z}$$

If we use Eq. (1.59) to find the voltage on the transmission line, we obtain

$$v(z, t) = \text{Re}\left\{Ae^{\alpha z}e^{j\beta z}e^{j\omega t} + Be^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right\}$$

$$= \left|A\right|e^{\alpha z}\cos(\omega t + \beta z + \phi_A) + \left|B\right|e^{-\alpha z}\cos(\omega t - \beta z + \phi_B)$$

By looking at this expression, we can understand the physical meaning of the real and imaginary parts of the complex propagation constant $\gamma$. The real part $\alpha$ represents attenuation and has units of Nepers per meter (Np/m). The imaginary part $\beta$ determines the wavelength of the wave, and is called the wavenumber with units of radians per meter (rad/m). $\beta$ is also called the spatial frequency or phase constant of the wave. The phase velocity of the wave is $u = \omega/\beta$ and the wavelength is $\lambda = 2\pi/\beta$ (Fig. 1.18).

![Figure 1.18: Propagating, attenuating forward wave.](image)

From Eq. (1.77), we can see that the constant $A$ represents the amplitude of the wave moving in the $-z$ direction, and $B$ represents the wave moving in the $+z$ direction. Because of this, we rename the constants so that $V_o^+ = B$ and $V_o^- = A$, so that (1.72) becomes

$$\tilde{V}(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

For a lossless line,

$$\gamma = j\omega\sqrt{LC} = j\beta$$  \hspace{1cm} \text{(Lossless line)}$$

so that the attenuation constant $\alpha$ is zero, and there is no decay of the amplitude of a wave as it propagates. The general solution for the voltage simplifies to

$$\tilde{V}(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$$  \hspace{1cm} \text{(Lossless line)}$$

Because many transmission lines can be approximated as lossless, we use this expression in most analyses instead of (1.72).
1.3.3 Current

To get the current on the line, we use one of the telegrapher’s equations in time-harmonic form:

\[- \frac{d\tilde{V}(z)}{dz} = (R + j\omega L)\tilde{I}(z)\]  

Substituting the general solution (1.72) leads to

\[
\tilde{I}(z) = -\frac{1}{R + j\omega L} \frac{d}{dz} \left[ V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \right] \]  

\[
= -\frac{1}{R + j\omega L} \left[ -\gamma V_o^+ e^{-\gamma z} + \gamma V_o^- e^{\gamma z} \right] \]  

\[
= \frac{\gamma}{R + j\omega L} \left[ V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z} \right] \]  

\[
= \sqrt{(R + j\omega L)(G + j\omega C)} \left[ V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z} \right] \]  

\[
= \frac{1}{Z_o} \frac{V_o^+}{v_o^+} e^{-\gamma z} - \frac{1}{Z_o} \frac{V_o^-}{v_o^-} e^{\gamma z} \]  

1.3.4 Reflection Coefficient

At the load end of a transmission line (Fig. 1.19), we can use boundary conditions to find the ratio of the forward and reflected waves.

![Figure 1.19: Transmission line and load.](image-url)
For the sinusoidal steady state, it is convenient to shift the coordinate system so that load end is at \( z = 0 \). The goal is to find

\[
\Gamma_L = \frac{V_o^-}{V_o^+}
\]  

(1.88)

The voltage and current boundary conditions at the load end, together with Ohm’s law for the load impedance, lead to the following relationship between the total current and voltage at the load end of the transmission line:

\[
\tilde{V}(0) = Z_L \tilde{I}_L(0)
\]  

(1.89)

Substituting the general voltage and current solutions leads to

\[
V_o^+ + V_o^- = Z_L \left( \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \right)
\]  

(1.90)

Now we can solve for the reflection coefficient:

\[
\Gamma_L = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}
\]  

(1.91)

Although this is the same expression as was obtained for the transient case (except that it is a phasor-domain formula and allows for complex load impedances), the meaning of the reflection coefficient is different. For the sinusoidal steady state, the forward and reverse waves exist everywhere on the transmission line. If we know \( V_o^+ \), for example, we can find \( V_o^- \) using \( \Gamma_L \), and then we can use Eq. (1.78) or (1.80) to find the voltage anywhere on the transmission line.

**Generalized reflection coefficient.** In the lossless case, it is also sometimes useful to define a generalized reflection coefficient as the ratio of the forward and reverse waves at any point on the transmission line:

\[
\Gamma(z) = \frac{V_o^- e^{j\beta z}}{V_o^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z}
\]  

(1.92)