1.5 Smith Chart

The Smith chart provides a graphical way to solve the transmission line equations that we have derived. Most high frequency engineering is done using computer aided design packages, so we don’t really need the Smith chart as a calculation tool. But it does provide a powerful way to communicate the behavior of a transmission line system visually. In fact, software packages and instruments often present computed or measured values on a Smith chart. Thus, the Smith chart is mainly a tool for gaining insight into transmission line systems.

The Smith chart is a plot of a reflection coefficient in the complex plane. Superimposed on that is a curved grid of lines that represent the load impedance corresponding to the reflection coefficient.

A passive load cannot reflect more power than is incident on it, so from Eq. (1.119), we must have \(|\Gamma_L| < 1\). Thus, for most transmission line systems the reflection coefficient is confined to the unit circle.

Now, let’s derive equations for the curved grid representing the impedance corresponding to \(\Gamma_L\). If we solve Eq. (1.91) for the load impedance, we obtain

\[
Z_L = Z_o \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{1.121}
\]

Because we don’t want to have to have a different Smith chart for every possible value of the characteristic impedance, we will rearrange this expression and work with normalized impedance, which we will identify with a lower case symbol:

\[
z_L = \frac{Z_L}{Z_o} = \frac{1 + \Gamma_L}{1 - \Gamma_L} \quad \text{(Normalized impedance)} \tag{1.122}
\]

Now, we break both \(z_L\) and \(\Gamma_L\) into their real and imaginary parts,

\[
r_L + jx_L = \frac{1 + \Gamma_{Lr} + j\Gamma_{Li}}{1 - \Gamma_{Lr} - j\Gamma_{Li}}
= \frac{1 + \Gamma_{Lr} + j\Gamma_{Li} (1 - \Gamma_{Lr} + j\Gamma_{Li})}{1 - \Gamma_{Lr} - j\Gamma_{Li} (1 - \Gamma_{Lr} + j\Gamma_{Li})}
= \frac{1 - \Gamma_{Lr}^2 - \Gamma_{Li}^2}{(1 - \Gamma_{Lr})^2 + \Gamma_{Li}^2} + j\frac{2\Gamma_{Li}}{(1 - \Gamma_{Lr})^2 + \Gamma_{Li}^2}
\]

With some algebra, the real and imaginary parts of this equation can be rearranged into the forms

\[
\left(\Gamma_{Lr} - \frac{r_L}{1 + r_L}\right)^2 + \Gamma_{Li}^2 = \left(\frac{1}{1 + r_L}\right)^2 \tag{1.123}
\]

\[
(\Gamma_{Lr} - 1)^2 + \left(\Gamma_{Li} - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2 \tag{1.124}
\]

Both of these equations represent circles. The first one is centered at \((\Gamma_{Lr}, \Gamma_{Li}) = (r_L/(1 + r_L), 0)\) and has radius \(1/(1 + r_L)\). The second circle is centered at \((1, 1/x_L)\) and has radius \(1/x_L\). Since the imaginary part of the impedance can be positive or negative, we have to consider two circles for each value of \(x_L\). For a given reflection coefficient in the complex plane, two circles intersect at that point, one given by (1.123) for a particular value of \(r_L\) and the other given by (1.124) for a value of \(x_L\). On the Smith chart, the \(r_L\) and \(x_L\) are labeled, so that the value of \(z_L = r_L + jx_L\) can easily be read from the chart.

Before using the Smith chart to analyze transmission lines, it is helpful to learn our way around the chart by considering some important landmarks (Fig. 1.27):
1. The unit circle $|\Gamma_L| = 1$ corresponds to lossless loads (capacitative, inductive, open circuit, short circuit).

2. The real axis corresponds to purely resistive load impedances. The left side of the real axis corresponds to $R_L < Z_0$, and the right side to $R_L > Z_0$.

3. The upper half plane represents inductive loads and the lower half plane represents capacitative loads.

4. The center of the Smith chart ($\Gamma = 0$) corresponds to a matched load ($z_L = 1$).

5. The point $\Gamma = -1$ corresponds to a short circuit.

6. The point $\Gamma = 1$ corresponds to an open circuit or infinite load impedance. Since small capacitances, large inductances, and large resistances all lead to $\Gamma \simeq 1$, all of the impedance circles go through that point.

![Smith chart landmarks](image-url)

Figure 1.27: Smith chart landmarks.
**Generalized reflection coefficient.** The next key principle for the Smith chart is that we can also plot the generalized reflection coefficient

\[
\Gamma(z) = \frac{V_o^+ e^{j\beta z}}{V_o^- e^{-j\beta z}} = \Gamma_L e^{j2\beta z}
\]

(1.125)

and the corresponding normalized input impedance

\[
z_{in}(z) = \frac{Z_{in}(z)}{Z_0}
\]

as a function of position on the line. As \(z\) moves away from zero, the phase of \(\Gamma(z)\) changes such that \(\Gamma(z)\) on the Smith chart moves around a circle centered at the origin. The radius of the circle is \(|\Gamma_L|\). Increasing \(z\) causes the phase angle in Eq. (1.125) to increase, which corresponds to counterclockwise rotation. Since the generator is at \(z = -\ell\), moving from the load towards the generator corresponds to clockwise rotation along the circle. If \(z\) changes by \(\lambda/2\), the generalized reflection coefficient moves once around the circle. If the distance moved along the line is given in wavelengths, so that \(L = \ell/\lambda\), then we rotate \(2L\) times around the circle.

**VSWR.** The VSWR on a transmission line is

\[
S = \frac{1 + |\Gamma|}{1 - |\Gamma|}
\]

(1.126)

where \(\Gamma\) can be the load reflection coefficient or the generalized reflection coefficient anywhere on the line. A circle of constant \(|\Gamma|\) on the Smith chart is also a circle of constant VSWR. When the VSWR circle crosses the positive real axis, the imaginary part of \(z_{in}\) is zero and the real part is equal to some value \(r > 1\), so the magnitude of the generalized reflection coefficient is

\[
|\Gamma(z)| = \frac{r - 1}{r + 1}
\]

(1.127)

If we solve this equation for \(r\) and compare the resulting expression to (1.126), we find that \(S = r\). As we move along a transmission line, the voltage standing wave maxima occur when the generalized reflection coefficient crosses the positive real axis. The minima occur when it crosses the negative real axis.

**Admittance.** When transmission lines or circuit elements are in parallel, it is convenient to convert from impedances to admittances. There are two ways to work with admittances on a Smith chart. One is to add another grid for admittances in a different color. The other is to shift an impedance point to a new point and then reinterpret the impedance circles on the Smith chart as admittance lines. The load admittance is

\[
y_L = \frac{1}{z_L} = \frac{1 - \Gamma_L}{1 + \Gamma_L}
\]

By comparing this to

\[
z_{in}(z = -\lambda/4) = \frac{1 + \Gamma_L e^{2j\beta z}}{1 - \Gamma_L e^{2j\beta z}}
\]

\[
= \frac{1 + \Gamma_L e^{j\pi}}{1 - \Gamma_L e^{j\pi}}
\]

\[
= \frac{1 - \Gamma_L}{1 + \Gamma_L}
\]

it can be seen that a \(\lambda/4\) rotation on the Smith chart transforms an impedance to an admittance. This corresponds to reflection with respect to the origin. So, if we are given an admittance, we can plot it on a single color Smith chart using its real and imaginary parts as values for the impedance circles, and the resulting reflection coefficient is found by reflecting the point about the origin.
1.5.1 Smith Chart Solution Procedure

To solve a lossless transmission line problem with generator and load impedance graphically on a Smith chart, the following steps are involved:

1. Find the normalized load impedance and plot it on the Smith chart.

2. Rotate the load reflection coefficient clockwise around a circle of constant radius on the Smith chart by an angle of $2\beta l$ or $2\ell/\lambda$ times around the circle.

3. Read the normalized input impedance $z_{in}(\ell)$ from the Smith chart, and unnormalize to get $Z_{in}$. This can be used in a voltage divider to get $V(-\ell)$, from which $V_0^+$ can be found using equations.

4. Other quantities can be read from the Smith chart as well:
   
   (a) VSWR: Real part of the input impedance when the constant VSWR circle crosses the positive real axis.

   (b) Voltage maxima/minima: The first voltage maximum occurs when the generalized reflection coefficient first crosses the positive real axis. The voltage minima occur when it crosses the negative real axis. The rotation angles at which the extrema occur can be read from the Smith chart and converted to distances along the line.
Example: Smith Chart Solution

Figure 1.28: Transmission line with source and load.

1. Find the normalized load impedance and plot on Smith chart.

2. Rotate to the generator end.

3. Read off $z_{in}$ and unnormalize to get $Z_{in}$.

4. Read the VSWR from the Smith chart.

5. Locate the first voltage minimum.