2.4 Maxwell’s Equations

Electromagnetic behavior can be described using a set of four fundamental relations known as *Maxwell’s Equations*. These equations are not derived, but rather are a mathematical model for observations. In general, these equations are given by

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_v \quad (2.35) \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad (2.36) \\
\nabla \cdot \mathbf{B} &= 0 \quad (2.37) \\
\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (2.38)
\end{align*}
\]

where

- \( \mathbf{E} \): Electric field intensity \ V/m
- \( \mathbf{H} \): Magnetic field intensity \ A/m
- \( \mathbf{D} \): Electric flux density \ C/m²
- \( \mathbf{B} \): Magnetic flux density \ Wb/m²
- \( \mathbf{J} \): Electric current density \ A/m²
- \( \rho_v \): Electric charge density \ C/m³

There is also an integral form of these equations that can be represented as

\[
\begin{align*}
\oint_S \mathbf{D} \cdot d\mathbf{s} &= \iint_V \rho_v \, dV = Q \quad (2.39) \\
\oint_C \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{s} \quad (2.40) \\
\oint_S \mathbf{B} \cdot d\mathbf{s} &= 0 \quad (2.41) \\
\oint_C \mathbf{H} \cdot d\mathbf{l} &= \int_A \mathbf{J} \cdot d\mathbf{s} + \frac{d}{dt} \int_A \mathbf{D} \cdot d\mathbf{s} \quad (2.42)
\end{align*}
\]

where \( S \) is the closed surface bounding the volume \( V \) and \( C \) is the closed path bounding the area \( A \). We will actually show how the integral and point forms of these equations are related a little later. For now, just take them on faith.

Suppose that the fields do not change in time (*static fields*). All of the time derivatives go to zero, and so Maxwell’s equations become

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= \rho_v \quad (2.43) \\
\nabla \times \mathbf{E} &= 0 \quad (2.44) \\
\nabla \cdot \mathbf{B} &= 0 \quad (2.45) \\
\nabla \times \mathbf{H} &= \mathbf{J} \quad (2.46)
\end{align*}
\]

Note that for the case of static fields, the electric and magnetic fields are no longer coupled. Therefore, we can treat them separately. For electric fields, we are dealing with *electrostatics*. For magnetic fields, we are dealing with *magnetostatics*.
2.4.1 Differential Forms

As differential forms, the field and source quantities of electromagnetics are given in Table 2.2.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Form</th>
<th>Type</th>
<th>Units</th>
<th>Vector/Scalar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electric Field Intensity</td>
<td>$E$</td>
<td>1-form</td>
<td>V</td>
<td>$E$</td>
</tr>
<tr>
<td>Magnetic Field Intensity</td>
<td>$H$</td>
<td>1-form</td>
<td>A</td>
<td>$\mathbf{H}$</td>
</tr>
<tr>
<td>Electric Flux Density</td>
<td>$D$</td>
<td>2-form</td>
<td>C</td>
<td>$\mathbf{D}$</td>
</tr>
<tr>
<td>Magnetic Flux Density</td>
<td>$B$</td>
<td>2-form</td>
<td>Wb</td>
<td>$\mathbf{B}$</td>
</tr>
<tr>
<td>Electric Current Density</td>
<td>$J$</td>
<td>2-form</td>
<td>A</td>
<td>$\mathbf{J}$</td>
</tr>
<tr>
<td>Electric Charge Density</td>
<td>$\rho$</td>
<td>3-form</td>
<td>C</td>
<td>$\rho_v$</td>
</tr>
</tbody>
</table>

Table 2.2: The differential forms of electromagnetics and their units and the corresponding vector quantities.

In terms of differential forms, Maxwell’s equations are

\[
\oint_P E = -\frac{d}{dt} \int_A B \\
\oint_P H = \frac{d}{dt} \int_A D + \int_A J \\
\oint_S D = \int_V \rho \\
\oint_S B = 0
\]  

(2.47)

With the laws expressed using differential forms instead of vectors, we can use differential form pictures to understand what they say about the behavior of the electromagnetic field.
2.5 Charge and Current Distributions

We need to remind ourselves about charges. The volume charge density is the amount of charge per unit volume. It can vary with position in the volume. The total charge is the integral of the charge density over the volume $V$, or

$$Q = \int_V \rho_v \, dV \quad (2.48)$$

where $Q$ is the charge in coulombs (C). When dealing with things like conductors, the charge may be distributed on the surface of a material. We therefore are interested in the surface charge density $\rho_s$ with units of C/m$^2$. This is the amount of charge per unit area on the surface. The total charge $Q$ would be the integral of $\rho_s$ over the surface. Finally, we can have a line charge density $\rho_\ell$ with units of C/m which is the amount of charge per unit distance along a line. An example of this might be a very thin wire. The total charge $Q$ would be the integral of $\rho_\ell$ over the length of the line segment.

By a similar token, we can discuss current density. $\mathbf{J}$ is the volume current density, measured in units of A/m$^2$. It represents the amount of current flowing through a unit surface area. The total current flowing through a surface $A$ is therefore

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} \quad (2.49)$$

We can also define a surface current density $\mathbf{J}_s$ with units of A/m. This means the current is confined to the surface, and so $\mathbf{J}_s$ is the amount of current per unit length, where the length represents the “cross-section” of the surface.

If we have a volume charge density $\rho_v$ moving at a velocity $\mathbf{u}$ (note that we are including the vector direction in the velocity), then $\mathbf{J} = \rho_v \mathbf{u}$.

2.6 Electric Fields and Flux

Let’s also quickly review electric fields and flux. Coulomb’s law states that

1. An isolated charge $q$ induces an electric field $\mathbf{E}$ at every point in space. At an observation point a distance $R$ from this charge, the electric field is

$$\mathbf{E} = \frac{\hat{R}}{4\pi\epsilon R^2} q$$

where $\hat{R}$ is the unit vector pointing from the charge to the observation point. $\epsilon$ is called the permittivity of the medium.

2. The force on a charge $q'$ due to an electric field is

$$\mathbb{F} = q' \mathbf{E} \quad (2.51)$$

The physical constant in Coulomb’s law is the permittivity of the material in which the charges are embedded, and is often given in terms of a relative permittivity with respect to the permittivity of free space:

$$\epsilon = \epsilon_0 \epsilon_r$$

$\epsilon_0 = 8.854 \times 10^{-12}$ F/m \hspace{1cm} permittivity of vacuum (free space)

$\epsilon_r$ \hspace{1cm} relative permittivity or dielectric constant of material

61
The electric flux density due to an electric field is

\[ \mathbf{D} = \varepsilon \mathbf{E} \]  

(2.52)

The electric field for a charge distribution is given by

\[ \mathbf{E} = \frac{1}{4\pi \varepsilon} \int_{v'} \rho' \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3} d\mathbf{v}' \]  

(2.53)

In cartesian coordinates this becomes

\[ \mathbf{E}(x, y, z) = \frac{1}{4\pi \varepsilon} \int_{v'} \rho'_{(x', y', z')} \frac{(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \, dx' \, dy' \, dz' \]  

(2.54)

where

\[ R' = x'\hat{x} + y'\hat{y} + z'\hat{z} \]  

(2.55)

\[ R = x\hat{x} + y\hat{y} + z\hat{z} \]  

(2.56)

\[ d\mathbf{v}' = dx' \, dy' \, dz' \]  

(2.57)

Even for a fairly simple charge distribution \( \rho'_{v} \), this equation is very complicated.

### 2.6.1 Differential Forms

With differential forms, Eq. (2.52) must be modified, because \( \mathbf{E} \) is a one-form, whereas \( \mathbf{D} \) is a two-form. In this case, we use the star operator:

\[ \mathbf{D} = \varepsilon \star \mathbf{E} \]  

(2.58)

This equation shows that tubes of electric flux \( \mathbf{D} \) are perpendicular to surfaces of \( \mathbf{E} \).
2.7 Review of Cylindrical and Spherical Coordinates

2.7.1 Cylindrical Coordinates

\[(r, \phi, z)\]
\[r = \sqrt{x^2 + y^2}\]
\[\phi = \tan^{-1}(y/x) = \text{angle from } +x \text{ axis}\]

<table>
<thead>
<tr>
<th>Swept Variable</th>
<th>Differential Length</th>
<th>Unit Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
<td>(dr)</td>
<td>(\hat{r})</td>
</tr>
<tr>
<td>(\phi)</td>
<td>(r , d\phi)</td>
<td>(\hat{\phi})</td>
</tr>
<tr>
<td>(z)</td>
<td>(dz)</td>
<td>(\hat{z})</td>
</tr>
</tbody>
</table>

2.7.2 Spherical Coordinates

\[(R, \theta, \phi)\]
\[R = \sqrt{x^2 + y^2 + z^2}\]
\[ \phi = \tan^{-1}(y/x) = \text{angle from } +x \text{ axis} \]
\[ \theta = \cos^{-1}(z/R) = \text{angle from } +z \text{ axis} \]

<table>
<thead>
<tr>
<th>Swept Variable</th>
<th>Differential Length</th>
<th>Unit Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( dR )</td>
<td>( \hat{R} )</td>
</tr>
<tr>
<td>( \theta )</td>
<td>( R , d\theta )</td>
<td>( \hat{\theta} )</td>
</tr>
<tr>
<td>( \phi )</td>
<td>( R \sin \theta , d\phi )</td>
<td>( \hat{\phi} )</td>
</tr>
</tbody>
</table>