Special Cases

Let’s consider a few common loads and line lengths, and for each we will determine the input impedance looking into the generator end of the line using

\[ Z_{in}(-\ell) = Z_o \frac{e^{-j\beta z} + \Gamma_L e^{j\beta z}}{e^{-j\beta z} - \Gamma_L e^{j\beta z}} \]  

or one of the alternate forms of this expression that we derived previously.

**Open circuit** \((\Gamma_L = 1)\):

\[
Z_{in}^{oc}(-\ell) = Z_o \frac{e^{j\beta \ell} + e^{-j\beta \ell}}{e^{j\beta \ell} - e^{-j\beta \ell}}
= Z_o \frac{2 \cos (\beta \ell)}{2 j \sin (\beta \ell)}
= -j Z_o \cot (\beta \ell)
\]  

**Short circuit** \((\Gamma_L = -1)\):

\[
Z_{in}^{sc}(-\ell) = j Z_o \tan (\beta \ell)
\]  

Notice that in both the open and short circuit cases, the input impedance is purely imaginary, corresponding to a reactive load, and the lines appear inductive or capacitative. By changing the length \(\ell\), we can make the line look like a capacitor or inductor of any value. This principle is often used in microwave designs. (Is it possible to realize a reactive impedance corresponding to a very large inductance or capacitance?)

**Half-integer line length** \((\ell = n\lambda/2, n = 1, 2, 3, \ldots)\)

\[
Z_{in}(-n\lambda/2) = Z_o \frac{Z_L + j Z_o \tan \beta \ell}{Z_o + j Z_L \tan \beta \ell} \bigg|_{\tan (\beta n\lambda/2) = \tan (n\pi) = 0}
= Z_L
\]  

The load impedance repeats along the line each half wavelength.

**Quarter-wave transformer** \((\ell = n\lambda/2 + \lambda/4, n = 0, 1, 2, \ldots)\) The input impedance looking into a quarter-wave section (or an integer number of half-wavelengths plus \(\lambda/4\)) as shown in Fig. 1.26 is

\[
Z_{in}(n\lambda/2 + \lambda/4) = Z_o \frac{Z_L + j Z_o \tan \beta \ell}{Z_o + j Z_L \tan \beta \ell} \bigg|_{\tan (\beta\lambda/4) = \tan (\pi/2) \to \infty}
= Z_L^2
\]  

In order to see a matched load looking into the quarter-wave line, we can set

\[
Z_{o1} = \frac{Z_{o2}^2}{Z_L} \Rightarrow Z_{o2} = \sqrt{Z_{o1} Z_L}
\]  

This is called quarter-wave matching.
1.4 Power

There are several ways to quantify power for sinusoidal steady state systems:

1. Instantaneous power: \( p^i(t) = v(t)i(t) \).
2. Time-average power: \( p_{av} = \frac{1}{T} \int_0^T p^i(t) \, dt, \quad T = \frac{2\pi}{\omega}. \)
3. Complex power: \( \tilde{P} = \tilde{V}\tilde{I}^* \)

It is easy to show that time-average power is related to complex power by
\[
p_{av} = \frac{1}{2} \text{Re}\{\tilde{V}\tilde{I}^*\}
\]
(1.113)

The imaginary part of \( \tilde{V}\tilde{I}^* \) is not associated with dissipated or supplied power, but rather represents changes in the amount of energy stored in inductive and capacitative elements. We will look at each of these for the case of a transmission line system.

1.4.1 Instantaneous Power

Energy is carried along a transmission line by both the forward and reverse waves. The instantaneous power arriving at a load is
\[
p^+(t) = v^+(t)i^+(t)
= \text{Re}\left\{V_o^+e^{j\omega t}\right\}\text{Re}\left\{\frac{V_o^+}{Z_o}e^{j\omega t}\right\}
= \text{Re}\left\{|V_o^+|e^{j\phi^+}e^{j\omega t}\right\}\text{Re}\left\{\frac{|V_o^+|}{Z_o}e^{j\phi^+}e^{j\omega t}\right\}
= \frac{|V_o^+|^2}{Z_o} \cos^2(\omega t + \phi^+)
\]
(1.114)

What this result means is that each half cycle of the forward wave delivers power to the load. If we repeat this derivation for the reverse wave, we find that
\[
p^-(t) = -\frac{|V_o^-|^2}{Z_o} \cos^2(\omega t + \phi^-)
= -|\Gamma_L|^2\frac{|V_o^+|^2}{Z_o} \cos^2(\omega t + \phi^+ + \theta_L)
\]
(1.115)
where $\Gamma_L = |\Gamma_L|e^{j\theta_L}$. The negative sign means that the reverse wave carries power away from the load. The net power delivered to the load is equal to the sum of the incident and reflected power:

$$p(t) = p^+(t) + p^-(t)$$ (1.116)

### 1.4.2 Time-Average Power

The time-average power associated with the forward wave is

$$p^+_{av} = \frac{1}{T} \int_{0}^{T} \frac{|V_o|^2}{Z_o} \cos^2 (\omega t + \phi^+) \, dt$$

$$= \frac{|V_o|^2}{Z_o} \frac{1}{2} \int_{0}^{T} \cos^2 (\omega t + \phi^+) \, dt$$

$$= \frac{|V_o|^2}{2Z_o}$$ (1.117)

The time-average power carried away by the reverse wave can be computed in the same way, but we will use the phasor expression in Eq. (1.113) to illustrate an alternate approach:

$$p^-_{av} = \frac{1}{2} \text{Re} \left\{ \hat{V}^- (0) \hat{I}^-(0) \right\}$$

$$= \frac{1}{2} \text{Re} \left\{ \hat{V}^- \left( \frac{V_o}{Z_o} \right)^* \right\}$$

$$= -\frac{|V_o|^2}{2Z_o}$$

$$= -|\Gamma_L|^2 \frac{|V_o|^2}{2Z_o}$$ (1.119)

The net time-average power delivered to the load is

$$p_{av} = p^+_{av} + p^-_{av} = \frac{|V_o|^2}{2Z_o} (1 - |\Gamma_L|^2)$$ (1.120)

What happens if the load is purely reactive (lossless)?

Another way to arrive at the same result is to compute the power absorbed by the load directly from the total phasor voltage at the load:

$$p_{av} = \frac{1}{2} \text{Re} \left\{ \hat{V}(0) \hat{I}(0) \right\}$$

$$= \frac{1}{2} \text{Re} \left\{ \hat{V}(0) \frac{\hat{V}^*(0)}{Z_L} \right\}$$

$$= \frac{1}{2} |\hat{V}(0)|^2 \text{Re} \left\{ \frac{1}{Z_L} \right\}$$

$$= \frac{1}{2} |V_o|^2 (1 + \Gamma_L)^2 \text{Re} \left\{ \frac{Z_L}{|Z_L|^2} \right\}$$

$$= \frac{|V_o|^2}{2|Z_L|^2} R_L |1 + \Gamma_L|^2$$

Although the expression appears different, it gives the same absorbed power as Eq. (1.120).