Chapter 1

Transmission Lines

A transmission line guides energy from one point to another in such a way that the energy does not spread as it propagates. Transmission line examples include

- Coaxial cables (network, television)
- Twisted pair lines (telephone, network)
- Waveguides, optical fibers
- Printed circuit board trace; metalized line on an integrated circuit
- Power lines
- Earth/ionosphere system

The circuit diagram symbol for a transmission line is two wires with junctions marked as small circles:

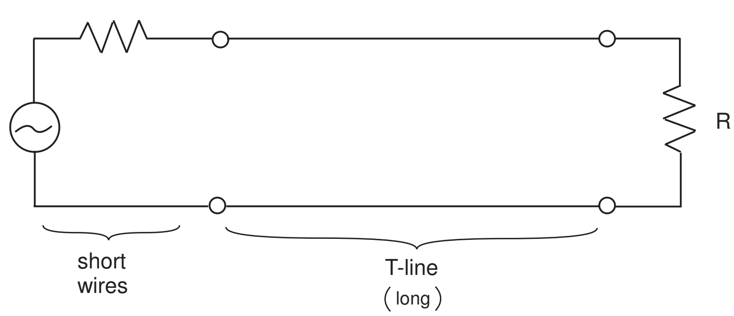

Figure 1.1: Circuit diagram symbol for a transmission line.

Circuit theory is an approximation to Maxwell's equations of electromagnetism. In a circuit, when the voltage between two nodes on a pair of wires is changed at one location, the voltage between any other pair of nodes on the same wires changes instantaneously. But in reality, the voltage change cannot propagate faster than the speed of light.

Transmission line theory can be viewed as a correction to circuit theory that is needed when wires or other conductors in a system are long enough that this propagation delay along the wires or conductors cannot be neglected.

One important aspect of transmission line theory is that many different types of lines, including systems that do not appear to be transmission lines at all, such as optical coatings and even empty space, can be treated using the same basic set of equations. We will study the behavior of transmission lines for two types of driving sources: transients or pulses, and sinusoidal or time harmonic excitation.

1.1 Transmission Line Equations

How can we analyze the behavior of currents and voltages on a transmission line? We need a set of equations that govern the currents and voltages at different locations along the line. One way to arrive at these equations is to model the transmission line as a sequence of lumped circuit elements. The inductance and capacitance provide the propagation delay as energy moves along the transmission line, and the resistance represents losses. One can then take the limit as the length of the lumped-element section goes to zero, and arrive at a set of partial differential equations for the current and voltage on the transmission line.

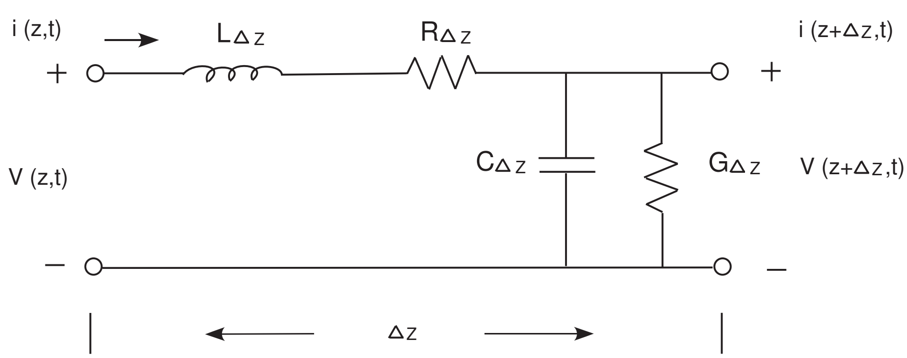

Figure 1.2: Lumped element model for a short section of a transmission line.

In the lumped element model, it is convenient to represent the capacitance, inductance, and resistance as per-unit-length quantities, so that

$$L = \text{Inductance per unit length (H/m)} \tag{1.1}$$
$$C = \text{Capacitance per unit length (F/m)} \tag{1.2}$$
$$R = \text{Resistance per unit length } (\Omega/\text{m}) \tag{1.3}$$
$$G = \text{Conductance per unit length (S/m)} \tag{1.4}$$

It is helpful at this point to understand these quantities in terms of a particular transmission line example. For a simple pair of parallel wires, $L\Delta z$ represents energy stored in the magnetic field around the wires for a section of length Δz. $C\Delta z$ represents the capacitance between the two pieces of wire. $R\Delta z$ represents the series resistance of the wires, and $G\Delta z$ represents parallel conductance of the dielectric material around the wires.

Lossless line. If the dielectric material between the conductors is a perfect insulator, then $G = 0$. If the conductors making up the transmission line are perfect, then $R = 0$. In this case, the line is said to be lossless. Real transmission lines are lossy, but in many cases the loss is small, so it is very common to approximate a transmission line as lossless.

1.1.1 Telegrapher Equations

Applying Kirchoff's voltage law (KVL) around the loop leads to

$$v(z,t) - i(z,t)R\Delta z - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z, t) = 0 \tag{1.5}$$

$$v(z,t) - v(z+\Delta z, t) = \Delta z \left[i(z,t)R + L\frac{\partial i(z,t)}{\partial t} \right] \tag{1.6}$$

$$-\left[\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} \right] = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{1.7}$$

When we let $\Delta z \to 0$, the left hand side becomes the definition of a derivative, so that

$$-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{1.8}$$

Using Kirchoff's current law at the top left node, we obtain

$$i(z,t) - G\Delta Z v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0 \tag{1.9}$$

Again taking the limit as $\Delta z \to 0$, this equation becomes

$$-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t} \tag{1.10}$$

Equations (1.8) and (1.10) are known as the Telegrapher equations.

1.1.2 Wave Equation

The Telegrapher equations are coupled first order partial differential equations. Since it is simpler to solve a single second order differential equation, we combine these two equations into a single equation. We first take the derivative of Eq. (1.8) with respect to z, to obtain

$$-\frac{\partial^2 v}{\partial z^2} = R\frac{\partial i}{\partial z} + L\frac{\partial^2 i}{\partial t \partial z} \tag{1.11}$$

We can substitute $\partial_t i$ from Eq. (1.10). We also need $\partial_{tz} i$, which we can get by differentiating Eq. (1.10) with respect to t:

$$-\frac{\partial^2 i}{\partial z \partial t} = G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} \tag{1.12}$$

Substituting Eqs. (1.10) and (1.12) into Eq. (1.11) leads to

$$-\frac{\partial^2 v}{\partial z^2} = -R\overbrace{\left[Gv + C\frac{\partial v}{\partial t}\right]}^{-\frac{\partial i}{\partial z}} - L\overbrace{\left[G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2}\right]}^{-\frac{\partial^2 i}{\partial t\,\partial z}} \tag{1.13}$$

$$\frac{\partial^2 v}{\partial z^2} = RGv + (RC + LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \tag{1.14}$$

This is a second order partial differential equation, where the only unknown is the voltage $v(z,t)$ on the transmission line. This is called the wave equation. A similar equation for $i(z,t)$ could be derived by eliminating the voltage instead, but once we know the voltage on the line, the current can be found using the transmission line equations.

1.1.3 Wave Solutions

How do we solve the wave equation (1.14)? Most of the time when we use differential equations in engineering, we look up or remember the general form of the solution and solve for the unknowns using initial or boundary conditions. For the wave equation, in the lossless case the general solution consists of two traveling waves of the form

$$v(z,t) = v^+(z - ut) + v^-(z + ut) \qquad \text{(Lossless line)} \tag{1.15}$$

The term $v^+(z-ut)$ represents a pulse or wave traveling to the right ($+z$ direction), and v^- represents a pulse traveling to the left ($-z$ direction). The functions v^+ and v^- depend on the excitation of the transmission line, and the constant u is determined by the coefficients of the wave equation.

Let's look at the first part of the general solution where the excitation produces a square pulse as given by

$$p(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

We will set $v^+(z - ut) = p(z - ut)$. At time $t = 0$, the pulse $v^+(z)$ is centered at $z = 0$. At the time $t = t_o$ the pulse becomes

$$v^+(z - ut_o) = \begin{cases} 1 & |z - ut_o| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

The pulse is now centered at the position $z = ut_o$. The pulse has moved in the $+z$ direction. The resulting velocity is given by

$$\text{velocity} = \frac{\Delta z}{\Delta t} = \frac{ut_o}{t_o} = u \tag{1.18}$$

The other part of the general solution $v^-(z + ut)$ travels at the same velocity in the $-z$ direction.

1.1.4 Phase Velocity

To solve for the constant u, we plug either part of the general solution (1.15) into the wave equation. Using the chain rule for the derivative,

$$\frac{\partial^2}{\partial z^2} v^+(z-ut) \quad = \quad v^{+\prime\prime}(z-ut) \tag{1.19}$$

where the primes denote ordinary differentiation. Follow the same process to get the second derivative with respect to time leads to

$$\frac{\partial^2}{\partial t^2} v^+(z-ut) \quad = \quad v^{+\prime\prime}(z-ut)(-u)^2 \tag{1.20}$$

Substituting these two terms into the wave equation gives

$$v^{+\prime\prime}(z-ut) \quad = \quad LC\ u^2\ v^{+\prime\prime}(z-ut). \tag{1.21}$$

In order for this equality to hold, we must have that $u^2 LC = 1$, so that

$$u = \frac{1}{\sqrt{LC}} \qquad \text{(Phase velocity)} \tag{1.22}$$

This quantity is called the phase velocity of waves on the transmission line.

For some common transmission lines, the phase velocity is

$$\text{Coaxial Cable}: \quad \frac{1}{\sqrt{\text{LC}}} \quad = \quad \left(\sqrt{\frac{2\pi}{\mu \ln(b/a)}}\right)\left(\sqrt{\frac{\ln(b/a)}{2\pi\epsilon}}\right) = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.23}$$

$$\text{Two Wire}: \quad \frac{1}{\sqrt{\text{LC}}} \quad = \quad \frac{1}{\sqrt{\mu\epsilon}} \tag{1.24}$$

$$\text{Parallel Plate}: \quad \frac{1}{\sqrt{\text{LC}}} \quad = \quad \frac{1}{\sqrt{\mu\epsilon}} \tag{1.25}$$

where μ and ϵ are parameters of the material separating the conductors. For these transmission lines, the velocity only depends on the properties of the material around the transmission line, not the geometry. For transmission lines which consist of a dielectric (insulator) and a pair of conductors, $\mu = \mu_o$, where μ_o is the permeability of free space ($\mu_o = 4\pi \times 10^{-7}$ H/m), and $\epsilon = \epsilon_r \epsilon_o$, where ϵ_o is the permittivity of free space ($\epsilon_o \simeq 8.854 \times 10^{-12}$ F/m) and ϵ_r is the relative permittivity of the dielectric. A typical value for the phase velocity is

$$u = \frac{1}{\sqrt{\mu\epsilon}} = \left(\frac{1}{\sqrt{\mu_o\ \epsilon_o}}\right)\left(\frac{1}{\sqrt{\epsilon_r}}\right) = \frac{c}{\sqrt{\epsilon_r}} \approx \frac{2}{3}c \tag{1.26}$$

where $c \simeq 3 \times 10^8$ m/s is the speed of light in a vacuum.