4.7 Maxwell’s Laws in Time-Harmonic Form

To go to sinusoidal steady state, we assume a time variation of $\cos \omega t$. Phasor notation is a very convenient way to work with sinusoidal waveforms. Recall that the definition of a phasor is

$$v(t) = \text{Re}\left\{ \tilde{V} e^{j\omega t} \right\} \tag{4.75}$$

where the phasor $\tilde{V}$ is a complex number. What we want to do is to express the components of the electric and magnetic fields as phasors.

Now, we often suppress the coordinate dependence of the fields, but all of the fields are functions of space and time:

$$E = E(x, y, z, t) = E(R, t) \tag{4.76}$$

$$E(R, t) = \text{Re}\left\{ \tilde{E}(R)e^{j\omega t} \right\} \tag{4.77}$$

In the phasor domain, time derivatives become multiplication by $j\omega$:

$$\frac{\partial E(R, t)}{\partial t} = \text{Re}\left\{ \tilde{E}(R)j\omega e^{j\omega t} \right\} \tag{4.78}$$

Therefore, Maxwell’s equations in time-harmonic (phasor) form are

$$\oint \tilde{E} \cdot d\ell = -j\omega \int \tilde{B} \cdot d\mathbf{s} \tag{4.79}$$

$$\oint \tilde{H} \cdot d\ell = j\omega \int \tilde{D} \cdot d\mathbf{s} + \int \tilde{J} \cdot d\mathbf{s} \tag{4.80}$$

$$\oint_S \tilde{D} \cdot d\mathbf{s} = \int \tilde{\rho}_v \, dV \tag{4.81}$$

$$\oint_S \tilde{B} \cdot d\mathbf{s} = 0 \tag{4.82}$$

In point form,

$$\nabla \times \tilde{E} = -j\omega \tilde{B} \tag{4.84}$$

$$\nabla \times \tilde{H} = j\omega \tilde{D} + \tilde{J} \tag{4.85}$$

$$\nabla \cdot \tilde{D} = \tilde{\rho}_v \tag{4.86}$$

$$\nabla \cdot \tilde{B} = 0 \tag{4.87}$$
Chapter 5

Plane Waves

We are now ready to look at the simplest form of electromagnetic waves.

5.1 Wave Equation

Instead of solving Maxwell’s equations directly to obtain wave solutions, we will transform the system of first order partial differential equations (PDEs) into a single second order PDE that is easier to solve. We start with Maxwell’s equations in time harmonic or phasor form,

\[ \nabla \times \mathbf{E} = -j\omega \mathbf{B} = -j\omega \mu \mathbf{H} \]
\[ \nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \rho_v \] (5.1)
\[ \nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J} = j\omega \epsilon \mathbf{E} + \mathbf{J} \]
\[ \nabla \cdot \mathbf{B} = \nabla \cdot \mu \mathbf{H} = 0 \] (5.2)

The goal is to eliminate all of the field quantities to get an equation for one field only.

**Conducting Media.** In order to handle lossy materials (conductors), we first rewrite Ampere’s Law. If we have a medium which has free charge allowing current flow, then \( \mathbf{J} = \sigma \mathbf{E} \), and

\[ \nabla \times \mathbf{H} = j\omega \epsilon \mathbf{E} + \sigma \mathbf{E} = j\omega [\epsilon + \sigma / j\omega] \mathbf{E} \]
\[ = j\omega \underbrace{[\epsilon - j\sigma / \omega]}_{\epsilon_c} \mathbf{E} \] (5.3)

(5.4)

This shows that in the phasor domain, the conductivity can be lumped together with the permittivity to produce a new effective complex permittivity:

\[ \epsilon_c = \epsilon - j\sigma / \omega = \epsilon_0 \left[ \epsilon_r - j \frac{\sigma}{\omega \epsilon_0} \right] = \epsilon_0 \epsilon_{cr} \] (5.5)

We also sometimes use the notation

\[ \epsilon_c = \epsilon' - j\epsilon'' \] (5.6)

for the real and imaginary parts of the complex permittivity. This reduces Ampere’s law for a conducting material into the form

\[ \nabla \times \mathbf{H} = j\omega \epsilon_c \mathbf{E} \] (5.7)
where $\epsilon_c$ is complex.

We will assume that there are no impressed sources in our region of interest (no sources inside the region of interest that are produced by external forces). We can still have charges that move in response to fields, or induced currents, but we have already taken those into account when we made $\epsilon_c$ into a complex number.

If we take the curl of Faraday’s law, we obtain

$$\nabla \times \nabla \times E = -j\omega \mu \nabla \times H$$

(5.8)

$$= -j\omega \mu (j\omega \epsilon_c E) = \omega^2 \mu \epsilon_c E$$

(5.9)

We now use the vector identity

$$\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E$$

(5.10)

where $\nabla^2 E$ is the Laplacian. In Cartesian coordinates:

$$\nabla^2 E = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}$$

(5.11)

This leads to

$$\nabla (\nabla \cdot E) - \nabla^2 E = \omega^2 \mu \epsilon_c E$$

(5.12)

From Gauss’ law: $\nabla \cdot \epsilon_c E = \epsilon_c \nabla \cdot E = 0$ since $\rho_v = 0$, so this equation simplifies to the **Homogeneous Wave Equation**:

$$\nabla^2 E + \omega^2 \mu \epsilon_c E = 0$$

(5.13)

This PDE is sometimes called the Helmholtz equation. If $\gamma^2 = -\omega^2 \mu \epsilon_c$, we call $\gamma$ the propagation constant. Therefore,

$$\nabla^2 E - \gamma^2 E = 0$$

(5.14)

Note that the magnetic field satisfies the same wave equation:

$$\nabla \times \nabla \times H = j\omega \epsilon_c \nabla \times E = j\omega \epsilon_c (-j\omega \mu H)$$

(5.15)

$$\nabla (\nabla \cdot H) - \nabla^2 H = \omega^2 \mu \epsilon_c H = -\gamma^2 H$$

(5.16)

$$\nabla^2 H - \gamma^2 H = 0$$

(5.17)

### 5.2 Lossless Media

Note that $J = \sigma E$ is like Ohm’s law $I = GV = V/R$, where $G$ = conductance. So, $\sigma > 0$ means energy will be dissipated (loss). If $\sigma = 0$, we call the material a **lossless medium**. In this case,

$$\gamma^2 = -\omega^2 \mu \epsilon_c = -\omega^2 \mu \epsilon$$

(5.18)

$$\gamma = j\omega \sqrt{\mu \epsilon} = jk$$

(5.19)

We call $k = \omega \sqrt{\mu \epsilon}$ the wavenumber. The units of $k$ are radians/meter. This is analogous to the transmission line quantity $\beta = \omega \sqrt{L/C}$, except that $\mu$ (H/m) and $\epsilon$ (F/m) are associated with a homogeneous space rather than a transmission line structure.